On First-Order Ordinary Differential Equations in Banach Spaces

By Madeaha Mabrouk Al-Ghanmi

A this thesis submitted for the requirements of the degree of Master of science (Mathematics - Differential equations)

Supervised By

Dr. Ezzat R. Hassan   Dr. Mohammed Sh. Alhuthali

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EXAMINATION COMMITTEE

	Name	Rank	Field	Signature
External Examiner	Prof. Dr. Gamal M. Mahmoud	Professor	Differential Equations for Dynamical Systems	
Internal Examiner	Dr.Sarah A. Al-AlSheikh	Assistant Professor	Differential equations	
Advisor	Dr.Ezzat R. Hassan	Assistant Professor	Differential equations	
Co-Advisor	Dr. Mohammed sh. Alhuthali	Associate Professor	Differential equations	

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Dedication

To my parents for their endless love, support and encouragement. There are not enough words to thank them or express my gratitude. I pray to God to keep them for me and provide them with health and wellness.

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Abstract

This thesis is mainly concerned with the question of the uniqueness of solutions of the Cauchy Problem

$$x' = f(t,x), \qquad x(t_0) = x_0, \tag{1}$$

where $f : [t_0, t_0 + L] \times E \to E$, $L > 0$, E is a Banach space. A necessary condition for non-uniqueness of solutions for the Cauchy problem, with the aid of an auxiliary scalar equation, is first established. As a consequence new uniqueness criteria are deduced. This result was previously known to hold only in the scalar case [65]. More precisely, Majorana [65] found out a very close relation between the number of the roots of a certain auxiliary scalar equation and those of the Cauchy Problem (1), where, $f : [t_0, t_0 + L] \times \mathbb{R} \to \mathbb{R}$. Our main approach is to retain this relation in a suitable generalized sense to provide an abstract version of Majorana's result in a finite dimension real (or complex) Banach space. We then generalize this result to infinitely dimensional real Banach spaces. These form our proper participation which are deferred until **Chapter 4** . The first part of the thesis introduces selected knowledge of basic facts of scalar Cauchy problems, calculus of abstract function, Cauchy problem in abstract spaces and theory of finite as well as infinite dimensional systems of differential equations is required. The presentation of these preliminaries chapters is self-contained and intends to convey a suitable starting point in this thesis.

Contents

Chapter 1

Introduction

In the language of mathematics, changing entities are called variables and the rate of change of one variable with respect to another a derivative. Equation which expresses a relationship among these variables and their derivatives are called differential equations. Cauchy Problem (whose name was coined by Jacques Hadamard in 1921) consists of a differential equation with initial conditions. Starting with the work of Peano in the 1890's, analysts have had a continuing interest in Cauchy problem (called, before Hadamard, the initial value problem IVP). Two major reasons for this interest, the first one is their prominent use as models of various phenomena. Secondly, the study of ordinary differential equations ODEs (related to Cauchy problem) acquire importance because partial differential equations PDEs can be transformed into (large) sets of (in some way equivalent) ordinary differential equation ODEs that are then solved by standard ODE solvers.

One of the crucial problems in the theory of Cauchy problems (CPs) was finding the conditions that guarantee the existence of solutions of that (CPs). This thesis contains results obtained from highly regarded sources. Although our purpose is to give a self-contained dissertation, the reader will not find the proofs of the fundamental results. The focus of this thesis is the Cauchy problem CP

$$x' = f(t,x), \qquad x(t_0) = x_0, \tag{1.1}$$

where $f : [t_0, t_0 + a] \times E \to E$, $a > 0$, E is $\mathbb{R}$ or an abstract space.
With such a problem, four fundamental questions arise:
(i) (Existence) Does a solution of the problem exist?.
(ii) (Locality) If the solution exists, what is its domain of existence?.
(iii) (Uniqueness) If a solution exists, is it unique?.
(iv) (Well-posedness) If the solution is unique, is the problem well-posed, i.e. is the solution in some sense a continuous function of the initial data?.

A large amount of our work in this thesis has been focused on the first three questions for the Cauchy problem for ordinary differential equation (1.1). A survey of the research history of this Cauchy Problem shows that a solution to (1.1) (a C^1- function $x(t) : [t_0, t_0 + L] \to E$ such that $x(t_0) = x_0$ and $x'(t) = f(t, x(t))$ i.e., it satisfies locally the related equation) always exists if $dimE < \infty$, L is small enough and $f(t,x)$ is continuous (this result is also known as Peano's theorem). The nature of solution is, however, quite different for differential equations with discontinuous right-hand sides. To convince ourselves that not every discontinuous ODE have a solution we give the following scalar Cauchy problem.

$$x' = 1 - 2sgnx, \qquad x(0) = 0,$$

or, equivalently

$$x' = \begin{cases} 3 & \text{if } x < 0, \\ -1 & \text{if } x > 0. \end{cases}$$

whose solution being given by

$$x(t) = \begin{cases} 3t + c_1 & \text{if } x < 0, \\ -t + c_2 & \text{if } x > 0. \end{cases}$$

We note that as t increases, each solution reaches the line $x = 0$. The direction field prevents the solutions from leaving the line $x = 0$ either upwards or downwards. Thus if the solution continued along this line, the function $x(t) = 0$ so obtained does not satisfy the equation in the usual sense since, for it $x(t) = 0$, and for $x = 0$ the right-hand side of the equation has the value $1 - sng0 = 1 \neq 0$. Although, this example has no solution in any interval containing 0, the following example has infinitely many solutions $x(t) = 1 + ct^2$,

$$x' = \frac{2(x-1)}{t}, \qquad x(0) = 1,$$

where c is an arbitrary constant. The treatment of such cases requires a generalization of the concept of solution.

On the other hand, Cauchy problems in infinite dimensional spaces may have no solutions. Showing the failure of Peano's theorem in infinite-dimensional Banach spaces is clearly a harder problem. Dieudonné [32] provided the first example of a continuous map from an infinitely dimensional Banach space c_0 for which there is no solution to the related Cauchy problem (1.1) (c_0 denotes the space of all real-valued sequences $x = (x_1, x_2, ...)$ with $x_n \to 0$ as $n \to \infty$ where $|x|_{c_0} = sup|x_n|$). This example seems to depend strongly on the properties of (c_0) (which is not reflexive). Many counterexamples in various infinite dimensional reflexive as well as nonreflexive Banach spaces followed, e.g., [39, 59, 93]. Afterwards, Godunov [39] proved that Peano's theorem is false in every infinite-dimensional Banach space. More precisely, for every infinite-dimensional Banach space E, $t_0 \in \mathbb{R}$, $x_0 \in E$, there exists a continuous mapping $f : \mathbb{R} \times E \to E$, such that there exists no solution to (1.1).

It turned out then that continuity was not the right assumption on the field f such that there is solution to the related Cauchy problem. Many celebrated works have been developed since the last century in order to obtain suitable extensions for the continuity notion on finitely dimensional spaces. Two main branches of the differential equations in Banach spaces were born. The topological structure of Banach spaces were the bases of this classification. Studying the Cauchy problem for differential equations in Banach space relative to the strong topology presents the first branch. In this branch compactness type conditions is imposed to guarantee only existence, this provides a first attempt to extend Peano's Theorem. The other approach in this branch is to utilize dissipative type conditions that assure existence and uniqueness of solutions, and the corresponding results extend Picard's theorem, i.e., the dissipative type conditions guarantee that a certain sequence of approximate solutions has at least a uniformly convergent subsequence. The second branch dealt with those problem in Banach space relative to the weak topology. In [22, 91, 53, 89] Peano's theorem is proved in the weak topology for Cauchy problem in reflexive Banach space. Really, weak topology appeared as a prototype to grapple with the lack of local compactness in infinitely dimensional Banach spaces. At least if the Banach space E is reflexive we recover locally compactness by endowing it with the weak topology. On the other hand, one of the attempts that appeared to generalize the previous results for nonreflexive Banach spaces was the so called measure of weak noncompactness for imposing weak compactness type conditions (dealing here with estimates for f involving a measure of noncompactness to, somehow, recover the locally compactness lost by the fact that the Banach space we are working on is no longer reflexive). As far as we know the first work in this direction is due to Cramer et al. [27]. Many researchers have improved and generalized results involving assumptions on the measure of weak noncompactness. Some of the progress in this direction are [19, 24, 25, 41].

The first two chapters of this thesis are assembled, in a coherent and comprehensive way, from results that are scattered across many different articles and books published over the last forty years. In what follows, we will give a description of each chapter of this thesis. Chapter 2 consists almost exclusively of theory of Scalar Cauchy problem. In Section 2.1, the well-known Peano's existence theorem is our starting point. There are many uniqueness theorems which apply, assuming a variety of conditions on the function f. A brief account of the uniqueness criteria is presented as well. The general uniqueness theorem due to E. Kamke which includes, as is well known, several special cases previously established by C. Carathéodory, M. Nagumo W. Osgood and Rogers [20, 75, 72, 83] is discussed. A generalization of Kamke's theorem is obtained by, among others, [1, 13, 55, 79]. While the uniqueness for the problem (1.1) has been investigated in many works, results dealing with nonuniqueness have been given by a few authors. Some of nonuniqueness results are derived by an inversion of uniqueness criteria of Kamke-type. Tamarkine [1] reverses the inequality in Osgood's condition and Lakshmikantham and Samimi's [1] nonuniqueness criteria are separately obtained by reversing Kamke -Perron's condition.

Majorana [65] followed an entirely different way to establish uniqueness criteria. First necessary condition for the nonuniqueness of (1.1) is established and a consequent new uniqueness criterion which does not require any special condition on the difference $f(t,x) - f(t,\overline{x})$. In fact, Majorana, by analyzing an auxiliary (not differential) equation, explored uniqueness criterion for (1.1). However, Majorana's results are not directly extendable to an arbitrary abstract space. This feature of Majorana's result gives us a strong motivation for achieving this generalization.

In the next section of this chapter we present Cauchy problem with discontinuous right hand side. The main results of the theory of differential equations with discontinuous right-hand sides are presented in this section. The treatment of such equations requires from the very start a generalization of the concept of solution. Carathéodory's solutions (as absolutely continuous functions which satisfy (1.1) almost everywhere) is proposed, and the conditions of their applicability are indicated. Uniqueness for discontinuous Cauchy problem is proved by using the concept of "Bounded Directional Variation". The consideration of differential equations with discontinuous right hand side is motivated by the fact that Majorana's result is satisfied for some discontinuous cases.

In the chapter 3, we consider Cauchy problem for ordinary differential equations in

Banach spaces. This chapter is organized as follows. In the first section, we present some topics from calculus of abstract function (by this we mean a function mapping an interval of the real line into a Banach) and also some theorem that enable us to apply this theory to differential equations in Banach spaces. Our aim in Section 3.2 is to provide, via the Cauchy problem for some equations of mathematical physics, the motivation to persuade people why abstract Cauchy problem?. Sections 3.3, 3.4 present a rich source of existence of weak solutions of differential equations in reflexive as well as nonreflexive Banach spaces. Strong solutions for Cauchy problems in abstract spaces are considered as well. We observe that weak topology coincides with strong topology in a Banach space E if, and only if, $dimE < \infty$. In the second section, we consider Cauchy problems in reflexive Banach spaces, the well-known Szep's theorem is the starting point in Cauchy problems in reflexive Banach spaces. On the other hand, when E is nonreflexive there are many results depend on using measure of weak noncompactness for imposing weak compactness type conditions. Thus, in chapter 3 we provide background material that consists principally of theory of Cauchy problem for ordinary differential equations in Banach spaces.

Chapter 4 is devoted to present our proper participation. We provide, in the first section of this chapter (section 4.1), our first participation in considering the space of all continuous anti-linear (or conjugate linear) functionals on a Banach space E to construct a scalar equation suitable to be in close relation with the Cauchy problem under consideration in finite dimensional Banach spaces. Our second attempt is to consider the Cauchy problem in infinite dimensional Banach spaces (section 4.2).

A wide variety of references are listed as well.

Chapter 2

Preliminary General Material on Scalar Cauchy Problem

We have tried to give a systematic exposition of the classical theory of scalar (CP), but no attempt at completeness can be made at this time, either in the text or in the references. All assertions in this chapter are made without proofs and the scope has been minimized to only material actually needed in the thesis.

First of all, by a scalar function we will always mean a real-valued function $x(t)$ of the real variable t. Let us consider a prototypical ordinary differential equation equipped with an initial condition (Cauchy problem),

$$x' = f(t, x), \qquad x(t_0) = x_0, \tag{2.1}$$

where $f : \overline{D} \to \mathbb{R}$, $\overline{D} = \{(t, x) : |t - t_0| \leq a,\ |x - x_0| \leq b\}$.

2.1 Classical Scalar Cauchy Problem

Throughout this section the function $f(t, x)$ is assumed to be continuous on $\overline{D}$.

Definition 2.1.1 *(Classical Solution). A solution of (2.1) is a function $x(t)$ continuously differentiable on an interval $I = \{t : |t - t_0| \leq \alpha < a\}$ (i.e., $\lim_{h\to 0} h^{-1}(x(t + h) - x(t))$ exists and is* *continuous) and satisfies*

(i) $x(t_0) = x_0$,

(ii) *for all* $t \in I$, $(t, x(t))) \in \overline{D}$,

(iii) $x'(t) = f(t, x(t))$ *for all* $t \in I$.

A well-known existence theorem due to Peano [78] says that a solution always exists in $|t - t_0| \leq \alpha$, $\alpha = min\{a, \frac{b}{M}\}$, $M = max_{(t,x)\in\overline{D}}|f(t, x)|$, if $f(t, x)$ is continuous.
Although, continuity of $f(t, x)$ is sufficient for the existence of a solution of (2.1), it dos not imply the uniqueness of the solutions as the following example shows.

Example 2.1.1 . *Let*

$$x' = x^{\frac{2}{3}}, \qquad x(0) = 0,$$

has an infinite number of solutions $x(t) = 0$,

$$x(t) = \begin{cases} 0 & \textit{if } 0 \leq t \leq c, \\ \frac{(t-c)^3}{27} & \textit{if } t > c. \end{cases}$$

Thus, to ensure the uniqueness, besides continuity, we need to impose some additional conditions on $f(t, x)$. Problems of this type are carefully analyzed in assuming that the variation of $f(t, x)$ with respect to x remains bounded. There are a lot of works devoted to investigating different types of the uniqueness conditions. As far as the authors know this activity has been started from Lipschitz's condition which represents one the of simplest forms:

$$|f(t, x) - f(t, \overline{x})| \leq L|x - \overline{x}|.$$

Clearly, if $f(t, x)$ satisfies Lipschitz condition, it is continuous relative to x, in this case it is then called Lipschitz continuous. The major advantage of Lipschitz's condition lies in the fact that it is easy to check. There are a huge number of papers in which Lipschitz-type condition has been generalized once and again. Some of the recent progress in this direction are contained in essence in[1]. We can quote here some outstanding generalizations of Lipschitz-type condition. Osgood's condition:

$$|f(t, x) - f(t, \overline{x})| \leq g(|x - \overline{x}|),$$

where $g(z)$ is a smooth function. Montel - Tonelli's condition:

$$|f(t,x) - f(t,\overline{x})| \leq h(t)g(|x - \overline{x}|),$$

where $h(t)$ is integrable. Nagumo's condition:

$$|f(t,x) - f(t,\overline{x})| \leq k|t - t_0|^{-1}|x - \overline{x}|, \ t \neq t_0 \ k \leq 1.$$

Regarding to the allowed values constant k, in contrast of Nagumo's condition, Krasnosel'skii - Krein achieved uniqueness result for (2.1) assuming that $k > 0$ as follows.

$$|f(t,x) - f(t,\overline{x})| \leq k|t - t_0|^{-1}|x - \overline{x}|, \ t \neq t_0 \ k > 0.$$

In their paper [55] Krasnosel'skii - Krein observed that all they needed to prove uniqueness is to impose an additional condition:

$$|f(t,x) - f(t,\overline{x})| \leq c|x - \overline{x}|^{\alpha}, \ c > 0, \ 0 < \alpha < 1, \ k(1-\alpha) < 1.$$

We also cite the work of Kooi [1], where he introduced a generalization of Krasnosel'skii - Krein condition as follows.

$$|t - t_0||f(t,x) - f(t,\overline{x})| \leq k|x - \overline{x}|, \ k > 0,$$

$$|t - t_0|^{\beta}|f(t,x) - f(t,\overline{x})| \leq c|x - \overline{x}|^{\alpha}, \ c > 0, \ 0 < \alpha < 1, \ k(1-\alpha) < 1 - \beta, \ \beta < \alpha.$$

Thomas Rogers [83] established a uniqueness criterion when $\frac{1}{t}$ in Nagumo's condition is replaced by $\frac{1}{t^2}$.

$$|f(t,x) - f(t,\overline{x})| \leq \frac{1}{t^2}|x - \overline{x}|, \ x \neq 0.$$

Kamke [50] assumed the existence of a suitable nonnegative function $g(t,|x|)$, with $g(t,0) = 0$, such that

$$|f(t,x) - f(t,\overline{x})| \leq g(t, |x - \overline{x}|).$$

The crucial point in applying Kamke theorem and some its generalization is that one has to study the new Cauchy problem:

$$u' = g(t,u), \quad u(0) = 0.$$

Reversing some of the above inequalities, together with slight change in conditions on $g(t, |x|)$, imply that the Cauchy problem (2.1) has more than one solutions (i.e., nonuniqueness criteria are deduced). We now cite a few results dealing with nonuniqueness criteria which have been given by this way. Tamarkine reverses the inequality in Osgood's condition to get nonuniqueness criterion

$$|f(t,x) - f(t,\overline{x})| \geq g(|x - \overline{x}|).$$

Separately, Lakshmikantham and Samimi' nonuniqueness criteria are obtained by, among other, reversing Kamke -Perron's condition

$$|f(t,x) - f(t,\overline{x})| \geq g(t, |x - \overline{x}|).$$

Here's a totally different way to investigating uniqueness criterion by analyzing a scalar auxiliary (not differential) equation, Majorana [65] developed a necessary condition for the nonuniqueness of (2.1), and as a consequence, a new uniqueness criterion is deduce. One of the major advantages of Majorana's result consists in the fact that any of the standard Lipschitz condition types do not apply. In the remaining of this section we present the most important result on which our participation in this thesis are based.

Majorana's result.

Majorana [65] explored, as mentioned above, another way to treat the question of uniqueness of solutions of (2.1) in $\mathbb{R}$. Majorana's idea is based on investigating a very close relation between (2.1) and a scalar (nondifferential) equation, namely,

$$u = (t - t_0)f(t, u) + x_0, \tag{2.2}$$

where u is the unknown and t a parameter whose role will be specified in the sequel. He establish, with the aid of this relation, a necessary condition for the nonuniqueness of the Cauchy problem (2.1). Let us cite the classical Majorana's theorems.

Theorem 2.1.1 *[65]. Let the function $f(t, x)$ be defined in $[t_0, t_0 + a] \times \mathbb{R}$, continuous with respect to t and such that $f(t, x_0) = 0$ for every $t \in [t_0, t_0 + a]$. Further let Cauchy problem (2.1)* *have two different classical solutions defined in* $t \in [t_0, t_0 + \alpha]$. *Then for every* $\varepsilon > 0$, *there exists* $t \in (t_0, t_0 + \alpha]$ *such that (2.2) has at least two different roots* u *with* $|u - x_0| < \varepsilon$.

An immediate consequence of this latter theorem is the following uniqueness criterion

Theorem 2.1.2 *[65]. Let the function* $f(t, x)$ *be defined in* $[t_0, t_0 + a] \times \mathbb{R}$, *continuous with respect to* t, *such that* $f(t, x_0) = 0$ *for every* $t \in [t_0, t_0 + a]$. *Further let there exist* $\varepsilon > 0$ *such that* $u = x_0$ *is the only root of (2.2) with* $|u - x_0| < \varepsilon$ *for every* $t \in [t_0, t_0 + \alpha]$. *Then, for (2.1),* $x(t) = x_0$ *is the only classical solution defined in* $t \in [t_0, t_0 + \alpha]$.

A crucial point in the previous theorems is the assumption $f(t, x_0) = 0$, that is, the assumption that (2.1) admits the constant solution. This restriction can be removed by means of the change of variable. If we know a solution φ of (2.1), then, we put $x = p + \varphi(t) - x_0$, (2.1) becomes

$$\begin{cases} p' = F(t, p), \\ p(t_0) = x_0, \end{cases} \tag{2.3}$$

where $F(t, p) = f(t, p + \varphi(t) - x_0) - f(t, \varphi(t))$. It is clear that any two different solutions of (2.1) are mapped to different solutions of (2.3). Moreover, (2.3) admits the constant solution, which corresponds to the solution φ of (2.1). And the scalar equation (2.2) is replaced by

$$u = (t - t_0)F(t, u) + x_0. \tag{2.4}$$

Majorana's theorems can be restated as follows.

Theorem 2.1.3 *[65]. Let the function* $f(t, x)$ *be continuous in* $[t_0, t_0 + a] \times \mathbb{R}$, *and let* $\varphi(t)$ *be a solution of (2.1). Assume further that the Cauchy problem (2.1) admit two different classical solutions defined in* $t \in [t_0, t_0 + \alpha]$. *Then for every* $\varepsilon > 0$, *there exists* $t \in (t_0, t_0 + \alpha]$ *such that (2.4) has at least two different roots* u *with* $|u - x_0| < \varepsilon$.

Theorem 2.1.4 *[65]. Let the function* $f(t, x)$ *be continuous in* $[t_0, t_0 + a] \times \mathbb{R}$, *and let* $\varphi(t)$ *be a solution of (2.1). Further let there exist* $\varepsilon > 0$ *such that* $u = x_0$ *is the only root of (2.4) with* $|u - x_0| < \varepsilon$ *for every* $t \in [t_0, t_0 + \alpha]$. *Then, for (2.1),* $x = \varphi$ *is the only classical solution defined in* $t \in [t_0, t_0 + \alpha]$.

Remark 2.1.1 *The second major advantage of Majorana's result lies in the fact that it is general enough to cover the classical case when $f(t,x)$ satisfies a Lipschitz condition with respect to x as well as the case when $f(t,x)$ is nonincreasing in x. Indeed, if the function $f(t,x)$ is Lipschitzian with respect to x (with Lipschitz constant L), then for any two solutions u_1 and u_2 of (2.4) corresponding to the same t, we have:*

$$|u_1-u_2| = (t-t_0)|F(t,u_1)-F(t,u_2)| = (t-t_0)|f(t,u_1+\varphi(t)-x_0)-f(t,u_2+\varphi(t)-x_0)| \leq L(t-t_0)|u_1-u_2|.$$

Hence $u_1 = u_2$ provided that $L(t-t_0) < 1$. It follows, by Theorem 2.1.6, that (2.1) has a unique solution for $(t-t_0) < \frac{1}{L}$. Therefore, Lipschitz uniqueness theorem can be considered as a corollary of Theorem 2.1.4.

Corollary 2.1.1 *Let the function $f(t,x)$ be continuous in $[t_0, t_0+a] \times \mathbb{R}$, such that $f(t,x_0) = 0$ for every $t \in [t_0, t_0+a]$. Further let $f(t,x)$ satisfy Lipschitz condition. Then, for (2.1), $x = x_0$ is the only classical solution defined in $t \in [0, \alpha]$.*

On the other hand, if $f(t,x)$ is continuous in $[t_0, t_0+a] \times \mathbb{R}$ and nonincreasing in x, then, according to Peano's uniqueness theorem , (2.1) has a at most one solution. Indeed, for $u_1 \leq u_2$,

$$0 \leq u_2 - u_1 = [(t-t_0)F(t,u_2)+x_0]-[(t-t_0)F(t,u_1)+x_0] = (t-t_0)[f(t,u_2+\varphi-x_0)-f(t,u_1+\varphi-x_0)] \leq 0.$$

We thus have $u_1 = u_2$. Then theorem 2.1.4 implies that (2.1) has at most one solution. Consequently, Peano's uniqueness theorem is recovered here as well.

Corollary 2.1.2 *Let the function $f(t,x)$ be continuous in $[t_0, t_0+a] \times \mathbb{R}$, such that $f(t,x_0) = 0$ for every $t \in [t_0, t_0+a]$. Further let $f(t,x)$ be nonincreasing with respect to x. Then, for (2.1), $x = x_0$ is the only classical solution defined in $t \in [t_0, t_0+\alpha]$.*

Remark 2.1.2 . *The third advantage of Majorana's result lies in the fact that it is quite natural to impose the continuity of $f(t,x)$ among the uniqueness conditions. It is worth noticing, in this connection, that Theorem 2.1.2 can also be applied to discontinue functions, as can be seen from* *the following simple example:*

Example 2.1.2 .

$$x' = f(t, x) = \begin{cases} \frac{1}{x - x_0} & \textit{if } x \neq x_0, \\ 0 & \textit{if } x = x_0, \end{cases}$$

$$x(0) = x_0.$$

The related scalar equation is $u = x_0$. We conclude that the given Cauchy problem satisfies the assumptions of Theorem 2.1.2 and hence, $x(t) = x_0$ is the only solution of the given CP.

2.2 Generalized Carathéodory Conditions for Discontinuous Cauchy Problem

In this section we review several existing results for the Cauchy problem (2.1) under the standing assumption that $f(t, x)$ is not required to be continuous. We face discontinuous differential equations in many applications. A large number of problems from mechanics and electrical engineering leads to differential equations with discontinuous right-hand sides because many physical laws are expressed by discontinuous functions, for example, a dry friction force of some electronic devices. Many differential equations appearing in control theory model objects with variable structure or with sliding motions that are discontinuous as well. The treatment of such equations requires from the very start a generalization of the concept of solution. Various definitions of the solutions of such equations are proposed in the literature, however, the best-known definitions of a solution of a differential equation with a discontinuous right-hand side are presented by Carathéodory. A solution of the Cauchy problem (2.1) in the sense of Carathéodory is defined as absolutely continuous functions $x : [0, a] \times \mathbb{R}^n$ which satisfy (2.1) almost everywhere, more precisely, if $x(t)$ satisfies

(i) $x(t_0) = x_0$,
(ii) $x'(t) = f(t, x(t))$ for almost all t with $|t - t_0| \leq a$. Or equivalently,

$$x(t) = x_0 + \int_0^t f(\tau, x(\tau))d\tau, \quad for \;\; almost \;\; all \quad t \in [0, a].$$

The well known example given, in the discontinuous case, is the following:

$$x' = sgnt, \qquad x(0) = 0,$$

whose solution being given by

$$x(t) = \begin{cases} t + c_1 & \text{if } t > 0, \\ -t + c_2 & \text{if } t < 0. \end{cases}$$

Proceeding from the requirement that the solution is continuous for $t = 0$, we obtain

$$x(0) = \lim_{t \uparrow 0} (-t + c_2) = \lim_{t \downarrow 0} (t + c_1).$$

Consequently, the solution is expressed by the formula $x(t) = |t|$. This example shows that a Carathéodory solution may not exist in any interval containing 0.

It turned out then that discontinuity of $f(t, x)$ was not the right assumption on $f(t, x)$ for the existence of classical solutions of (2.1). As Peano's theorem is considered to be the main result in the classical scalar differential equation with continuous right hand side, Carathéodory's standard theorem [20] is the main result in discontinuous differential equation, we thus give here the existence as well as the uniqueness results for ODEs, whose right-hand side is measurable in t and continuous in x, which has been considered by Carathéodory.

Theorem 2.2.1 *[20] (Carathéodory theorem). Let the $f : [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$ be a bounded function.* *(i) (Carathéodory conditions) If the map $t \mapsto f(t, x)$ is Lebesgue measurable for each x and the map $x \mapsto f(t, x)$ is continuous for almost all $t \in [0, a]$, then the Cauchy problem (2.1) has at least one solution.*

(ii) *If the map* $t \mapsto f(t,x)$ *is Lebesgue measurable for each* x *and the map* $x \mapsto f(t,x)$ *is Lipschitz continuous, for almost all* $t \in [0,a]$*, with a uniform Lipschitz condition, i.e.,*

$$|f(t,x) - f(t,\overline{x})| \leq L(t)|x - \overline{x}|,$$

where $L(t)$ *is a Lebesgue integrable function on* $[0,a]$*. Then the Cauchy problem (2.1) has at most one solution.*

We notice that if $f(t,x)$ satisfies Carathéodory conditions, then, the superposition $f(.,x(.))$ is Lebesgue integrable for measurable function $x(t)$. Many celebrated works have been developed in order to guarantee the existence or the uniqueness of local solutions of discontinuous scalar Cauchy problem. The continuity in Carathéodory conditions is modified to quasi-monotonicity [34], namely,
(C) for almost all $t \in [0,a]$, all $x \in R$,

$$\limsup_{y \uparrow x} f(t,y) \leq f(t,x) \leq \liminf_{y \downarrow x} f(t,y).$$

Conditions along the lines of quasi-increasing, Wu [92] started in 1964

$$\lim_{y \uparrow x} f(t,y) \leq f(t,x) = \lim_{y \downarrow x} f(t,y).$$

In 1993 Biles [8] improved the foregoing to

$$\limsup_{y \uparrow x} f(t,y) \leq f(t,x) = \lim_{y \downarrow x} f(t,y).$$

In 1996 [86] this has been relaxed to

$$\limsup_{y \uparrow x} f(t,y) \leq f(t,x) = \liminf_{y \downarrow x} f(t,y).$$

It turned out that the boundedness and measurability of $f(t,x)$ are quite natural and thus are usually assumed in the subject, the literature differs on the continuity condition, so several authors have improved Carathéodory result by weakening the continuity as mentioned just above.

Conclusion: Basically two branches were born in the theory of discontinuous differential equations, the first, explicitly, measurability of the superposition $f(., x(.))$ for every (AC) function x, is required see Biles and Binding [11] among others. In the second the function $f(t, x)$ is supposed to satisfy collections of conditions guaranteeing the measurability of the superposition $f(., x(.))$ see,among others, Carathéodory, Wu, Biles, Rzymowski, Walachowski and references therein.

Later on Lipschtiz continuous of $f(t, x)$ relative to x is improved using the notions of directional continuity and of bounded directional variation as shown in the following section.

2.3 System of Discontinuous Scalar Cauchy Problem

In this section, let $f(t, x) : \mathbb{R}^{n+1} \to \mathbb{R}^n$. For a fixed $M > 0$, consider the cone:

$$\Gamma^M = \{(t, x) \in \mathbb{R}^{n+1} | \|x\| \leq Mt\}.$$

By means of such a cone Γ^M one can define a relation of partial ordering on R^{n+1} as follows:

$$(s, y) \preceq (t, x) \;\; iff \;\; \|y - x\| \leq (t - s).$$

Using the order relation generated by the cone Γ^M the concept of boundedness directional variation.

Definition 2.3.1 *(Bounded Variation). The function $f(t, x)$ has bounded Γ^M- variation, if there exist constants $L < M$ such that*

$$\sum_{i=1}^{n} |f(t_i, x(t_i)) - f(t_{i-1}, x(t_{i-1}))| \leq L,$$

for every finite sequence (t_i, x_i), $i = 0, 1, ..., N$, with $(t_0, x_0) \preceq (t_1, x_1) \preceq ... \preceq (t_N, x_N)$.

Definition 2.3.2 *(Locally Bounded Variation). The function $f(t,x)$ has locally bounded Γ^M-variation, if for every $(t_0,x_0) \in \mathbb{R}^{n+1}$ there exist $\delta > 0$ and a constant L such that*

$$\sum_{i=1}^{n} |f(t_i, x(t_i)) - f(t_{i-1}, x(t_{i-1}))| \le L,$$

for every finite sequence (t_i, x_i), $i = 0, 1, ..., N$, with $(t_0, x_0) \preceq (t_1, x_1) \preceq ... \preceq (t_N, x_N)$, $t_N < \delta + t_0$.

The following existence and uniqueness result is proved in [15].

Theorem 2.3.1 *[15]. If $f(t,x) : \mathbb{R}^{n+1} \to \mathbb{R}^n$ has locally bounded Γ^M- variation and $|f(t,x)| \le C < M$ for all (t,x). Then the appropriate Cauchy problem expressed by (2.1) has a unique solution defined in $[t_0, \infty)$.*

The autonomous Cauchy problem

$$x'(t) = f(x(t)), \qquad x(t_0) = x_0, \tag{2.5}$$

has been studied in assuming f to be Lipschitz continuous, a classical theorem states the existence and uniqueness of solutions of (2.5) [43, 59].
On the other hand, Bressan [18] proved, with the aid of directional continuity concept defined in terms of a cone on $\mathbb{R}^n$, uniqueness results.

By a cone we mean a closed set $\Gamma \subset \mathbb{R}^n$ such that

$$\lambda_1 x_1 + \lambda_2 x_2 \in \Gamma \;\; whenever \;\; x_1, x_2 \in \Gamma, \;\; \lambda_1, \lambda_2 \ge 0.$$

A cone Γ determines a relation of partial ordering among points in R^n, defined as

$$x \preceq y \;\; iff \;\; y - x \in \Gamma.$$

A directionally continuous concept is defined in terms of the order relation generated by the cone Γ as well.

Definition 2.3.3 *[16]. We say that the map $f : \mathbb{R}^n \to \mathbb{R}^n$ is directionally continuous if at each point $x \in \mathbb{R}^n$ one has*

$$\lim_{y \to x, y - x \in \Gamma} f(y) = f(x).$$

Using the order relation generated by the cone Γ the concept of boundedness directional variation.

Definition 2.3.4 . *We say that the map $f : \mathbb{R}^n \to \mathbb{R}^n$ has bounded directional variation in the direction of the cone Γ if*

$$\sup \sum_{i=1}^{N} |f(x_i) - f(x_{i-1})| < \infty,$$

where the supremum is taken over all finite sequences of points $x_0, x_1, ..., x_N \in \mathbb{R}^n$, $N \geq 1$, such that $\sum_{i=1}^{N} |x_i - x_{i-1}| \leq 1$, $x_0 \preceq x_1 \preceq ... \preceq x_N$.

Theorem 2.3.2 *[18]. Let $f(x) : \mathbb{R}^n \to \mathbb{R}^n$ be bounded, possibly discontinuous, and let $g : [0, T] \to \mathbb{R}^n$ be bounded and measurable. Assume that there exists a cone Γ with the following properties. The map f has bounded directional variation in the direction of the cone Γ. Moreover, there exist constants $0 < \zeta < \rho$ such that*

$$B(f(x), \rho) \subseteq \Gamma, \quad x \in \mathbb{R}^n$$

$$|g(t)| < \zeta, \quad t \in [0, T].$$

Then, the Cauchy problem:

$$x' = f(x) + g(t), \quad x(0) = 0,$$

has a unique Carathéodory solution, defined for all $t \in [0, T]$.

2.4 Infinite-Order System of Cauchy Problems

We desire in this section develop a similar result to infinite system of Cauchy Problems

$$x'_\alpha = f_\alpha(t, ..., x_\alpha, ...), \qquad x_\alpha(0) = 0, \tag{2.6}$$

where $f : [0,1] \times \mathbb{R}^N \to \mathbb{R}^N$, α runs through some arbitrarily set N (not necessarily countable). Throughout this section the space $\mathbb{R}^N$ is equipped with the product topology which means that any function $x(t) = (x_\alpha(t), x_\beta(t), ...)$ taking values in $\mathbb{R}^N$ is continuous if and only if each of its component $x_\alpha(t)$ is continuous, thus, if N is finite, $\mathbb{R}^N$ is just the usual Euclidean space considered in the previous section. By a solution of (2.6) we shall mean a continuous function $x = (x_\alpha, x_\beta, ...) : [0,1] \to \mathbb{R}^N$ that satisfies $x_\alpha(0) = 0, \alpha \in N$, and such that each component function $x_\alpha : [0,1] \to \mathbb{R}$ is absolutely continuous and satisfies $x'_\alpha(t) = f_\alpha(t, ..., x_\alpha(t), ...)$ almost everywhere in $[0,1]$.

We note that all mathematical concepts, lim's, sup's, inf's, inequalities ... etc, are interpreted componentwise.

For x and y are in $\mathbb{R}^N$ we write $x \leq y$ if $x_\alpha \leq y_\alpha$, for all $\alpha \in N$. Carathéodory's result [20] for Cauchy problem (2.6) has been generalized once and again, especially in the scalar case, see, [8, 9, 10, 11, 81]. Conditions along the lines of quasi-monotonicity, see [8, 10, 11, 86], were greatly improved by [34, 81].

Theorem 2.4.1 *[10]. Let $f : [0,1] \times \mathbb{R}^N \to \mathbb{R}^N$ satisfy the following conditions:*
(i) *(Measurability) for all $x \in C\big([0,1], \mathbb{R}^N\big)$, $f(., x(.)) : [0,1] \to \mathbb{R}^N$ is Lebesgue measurable,*
(ii) *(Quasimonotonicity) for all $t \in [0,1]$ and all $x \in \mathbb{R}^N$, $f(t,x)$ is quasincreasing, i.e., for all x,$y \in \mathbb{R}^N$ with $x_\alpha \leq y_\alpha$ for all $\alpha \in N$, then*

$$f_\beta(t,x) \leq f_\beta(t,y) \ \ for \ \ all \ \ \beta \in N \ \ with \ x_\beta = y_\beta.$$

(iii) *(Boundedness) for all $x \in \mathbb{R}^N$ and almost all $t \in [0,1]$, $||f(t,x)|| \leq M(t)$, where $M(t)$ is an integrable function.*
(iv) *(Quasisemicontinuity) for each $\alpha \in N$ and almost all $t \in [0,1]$, $f(t,.)$ is quasisemicontinuous, i.e., for each $\zeta \in N$ and $x \in \mathbb{R}^N$ we have*

$$\begin{aligned} &\limsup_{y \uparrow x_\zeta} f_\zeta(t, x_\alpha, ..., y, ...) \leq f_\zeta(t, x_\alpha, ..., x_\zeta, ...) \\ &\leq \liminf_{y \downarrow x_\zeta} f_\zeta(t, x_\alpha, ..., y, ...), \end{aligned} \tag{2.7}$$

Then CP expressed by (2.6) has extremal solutions defined almost everywhere in $[0,1]$.

Chapter 3

Preliminary General Material on Abstract Cauchy Problem

As far as we all know, the concept of the differential equation governing evolution in abstract spaces has been suitably formulated since the 70s. Brief orientation about main content in this chapter: In Section 3.1, we assemble the preliminary material providing the necessary tools including the calculus for abstract maps relevant to the later development. Sections 3.2, 3.3 and 3.4 are devoted to the investigation of the fundamental theory of ODEs such as existence and uniqueness, global existence under the continuity assumption for ODEs.

3.1 Banach Spaces and Differential Calculus

Necessary concepts that are assumed to be known: Vector space(a set that is closed under finite vector addition and scalar multiplication). Normed space (vector space with the additional structure of a norm). Metric space (a set with a global distance function). Normed spaces are metric spaces: the metric in this case is derived from a norm. The metric space E is said to be complete if every Cauchy sequence in E has a limit which is an element of E. When the induced metric space is complete, the normed spaces is called Banach space. Continuous linear functional on E is a linear mapping to the underlying field of E. Reflexive Banach space, also known as regular Banach space, E is one such that, for every continuous linear functional F on the conjugate space E^* (the space of all linear functional f defined on E), there corresponds a point x_0 of E such that $F(f) = f(x_0)$ for each element $f \in E^*$, i.e., under this mapping E is isometric with E^{**} (the space of all linear functional F defined on E^*). A necessary condition for E to be reflexive is that each linear continuous functional attains its norm. A normed space $(E, \|\|_E)$ generates a topology, known as strong topology or norm topology. A sequence x_n in E converges strongly to a limit $x \in E$ if and only if $\|x_n - x\| \to 0$ as $n \to \infty$. In some cases it is useful to work in topologies on vector spaces that are weaker than a norm topology, the norm topology on infinite-dimensional spaces is so strong that very few sets are compact or pre-compact in these topologies, making it difficult to apply compactness methods in these topologies, instead, one often works in a weaker topology, in which compactness is easier to establish in infinite-dimensional spaces. The weak topology on E is the topology $\sigma(E, (f)_{f \in E^*})$ that makes all the f's continuous. The weak topology is weaker than the norm topology, i.e., every weakly open (resp. closed) set is strongly open (resp. closed). A sequence x_n in E converges weakly to a limit $x \in E$ if and only if for every $f \in E^*$ $\lim_{n\to\infty} (f, x_n) = (f, x)$. Strong convergence implies weak convergence with the same limit (the converse is not generally true). If $dimE < \infty$, then weak convergence implies strong convergence. By an abstract function $\varphi : I \to E$ we mean a function mapping an interval of the real line into a Banach space. The concepts of continuity and differentiability of an abstract function are defined as follows

Definition 3.1.1 . *An abstract function $x(t)$ is said to be*
(i) continuous at the point $t_0 \in I$, if

$$\lim_{t \to t_0} \|x(t) - x(t_0)\| = 0,$$

if $x : I \to E$ is continuous at each point of I, then we say that x is continuous on I and we write $x \in C[I, E]$.

(ii) Lipschitz continuous on I with Lipschitz constant L if

$$\|x(t_1) - x(t_2)\| \leq L|\ t_1 - t_2\ |, \quad for\ all\ \ t_1, t_2 \in I.$$

Definition 3.1.2 . *Let $I \subseteq \mathbb{R}$ be an interval. A function $x : I \to E$ is said to be strongly continuous on I if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\sum_{i=1}^{n} \|x(t_i) - x(s_i)\| \leq \varepsilon$$

for every finite number of nonoverlapping intervals (s_i, t_i), $i = 1, ..., n$, with $(s_i, t_i) \subset I$ and

$$\sum_{i=1}^{n} |t_i - s_i| \leq \delta.$$

Definition 3.1.3 *The (strong) derivative of an abstract function $x(t)$ is defined by*

$$\lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t},$$

where the limit is taken in the strong sense, that is,

$$\lim_{h \to 0} \| \frac{x(t + h) - x(t)}{h} - x'(t)\| = 0,$$

Definition 3.1.4 . *An abstract function $x(t)$ is said to be weakly continuous (weakly differentiable) at the point $t_0 \in I$, if for every $\Gamma \in E^*$ the scaler function $\Gamma[x(t)]$ is continuous (differentiable) at t_0.*

The image of $x \in E$ under $x^* \in E^*$ will be denoted by $< x, x^* >$. For each $x \in E$, $K(x)$ denotes the nonempty set of all $x^* \in E^*$ for which $< x, x^* >= \|x\|^2 = \|x^*\|^2$. A duality mapping of E is a function $k : E \to 2^{E^*}$ such that $k(x) \in K(x)$ for each $x \in E$. We shall rely on the following simple observation due to [7, 51].

Suppose that $(C1), (C2)$ below are fulfilled

(C1) $x(t)$ has a weak derivative $x'(t) \in E$,

(C2) $\|x(t)\|$ is differentiable. Then

$$\|x(t)\| \frac{d}{dt} \|x(t)\| = Re < x'(t), x_t^* >, \tag{3.1}$$

for each $x_t^* \in K(x(t))$.

Definition 3.1.5 *(An algebra). A set E is an algebra over the scalar field F if E is a linear space over F where multiplication is also defined between the elements of E satisfying the following properties:*
(i) *to every ordered pair $(x, y) \in E$ there corresponds a unique element xy in E;*
(ii) *(distributivity) $(x + y)z = xz + yz$ and $x(y + z) = xy + xz$;*
(iii) *(associativity) $(xy)z = x(yz)$;*
(iv) $\alpha x \beta y = \alpha\beta xy$, $x, y \in E$, $\alpha, \beta, \in F$.

Definition 3.1.6 *(Banach algebra). A Banach algebra over a field F is an algebra E over the field F such that $||xy|| \leq ||x|| \, ||y||$.*

3.2 The Cauchy Problem for Some Equations of Mathematical Physics: The Abstract Cauchy Problem

This section is understood as motivation for our central purpose of well posed Cauchy problem in Banach space: several equations of mathematical physics can be transformed into equivalent abstract Cauchy problem (Cauchy problem in abstract spaces). We deal here with the heat-diffusion equation in a two-dimensional square

$$\frac{\partial u}{\partial t} = \kappa \Delta u = \kappa(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}), \tag{3.2}$$

κ is a positive constant, $t \geq 0$, $\Omega = \{(x, y)|0 < x, y < \pi\}$ with initial-boundary condition

$$u(x, y, 0) = u_0(x, y) \;\; \{(x, y)|0 < x, y < \pi\}, \quad u(x, y, t) = 0 \;\; \{(x, y) \in \Gamma, t \geq 0\}, \tag{3.3}$$

where Γ is the boundary of Ω. It is customary to call $u(x, y, t)$ a classical solution of (3.2) and (3.3) if u is continuously differentiable up to the degree required by the equation that satisfies identically (3.2) and (3.3)- that is u is continuous in $\overline{\Omega} \times [0, \infty)$, u_t exists and is continuous in the same region, u_x, u_y, u_{xy}, u_{xx}, u_{yy} exist in $\Omega \times [0, \infty)$ and are continuous in $\overline{\Omega} \times [0, \infty)$ such that (3.2) and (3.3) are satisfied. We can write the initial-boundary value problem (3.2) and (3.3) as a pure initial value problem for an ordinary differential equation in a Banach space as follows. Let us denote $E = C_\Gamma(\overline{\Omega})$ the space of all continuous functions in $\overline{\Omega}$ that vanish at the boundary Γ, endowed with the supremum norm. Define an operator A as follows:

$$Au = \kappa(u_{xx} + u_{yy}), \tag{3.4}$$

where $D(A)$ consists of all $u \in E$ such that u_x, u_y, u_{xy}, u_{xx}, u_{yy} exist in Ω and are continuous in $\overline{\Omega}$ and $\Delta u \in E$ (that is, $\Delta u = 0 on \Gamma$). Now let u be a classical solution of (3.2) and (3.3) in the sense made precise above. Define a function $t \mapsto u(t) \in E$ for $t \geq 0$ as follows:

$$u(t)(x,y) = u(x,y,t), \qquad t \geq 0, \qquad (x,y) \in \overline{\Omega}. \tag{3.5}$$

Clearly, $u(t) \in E$ for all $t \geq 0$. On the other hand, if we define $v(t) \in E$ by:

$$v(t)(x,y) = u_t(x,y,t), \qquad t \geq 0, \qquad (x,y) \in \overline{\Omega},$$

it follows, by uniform continuity of u_t on compact subsets of $\overline{\Omega} \times [0,\infty)$, that $v(.)$ is continuous in $t \geq 0$. Moreover, if $t > 0$, we have, by virtue of an elementary application of the mean value theorem and of the continuity of u_t

$$\lim_{h\to 0} \|h^{-1}(u(t+h) - u(t)) - v(t)\| = \lim_{h\to 0} \sup_{(x,y)\in\Omega} |h^{-1}(u(x,y,t+h) - u(x,t,t)) - u_t(x,y,t)| = 0.$$

It turns out that $u(.)$ is continuously differentiable in the sense of the norm of E for $t > 0$ and

$$u' = Au, \qquad u(0) = u_0(u_0(x,y)). \tag{3.6}$$

Conversely, let $t \mapsto u(t) \in E$ be a function defined and continuously differentiable in $t \geq 0$, and such that $u(t) \in D(A)$ and satisfies (3.6). Then it is rather easy to show that

$$u(x,y,t) = u(t)(x,y), \qquad t \geq 0, \qquad (x,y) \in \Omega,$$

is a classical solution of (3.2) and (3.3) with $u_0(x,y) = u(0)(x,y)$ and $u(x,y,t) = 0$ $((x,y) \in \Gamma, t \geq 0)$. This shows the complete equivalence of our original initial-boundary value problem and the initial value or Cauchy problem (3.6). It should be noted that the existence and uniqueness of solutions for the Cauchy problem (3.6) is strongly related to the theory of one parameter semigroup and thus is omitted here. Taking as motivation the observations in this section most of our efforts in this thesis are concentrated in the study of the classical Cauchy problem (CP)

$$x' = f(t, x), \qquad x(0) = x_0, \tag{3.7}$$

where $f : [0, a] \times E \to E$, E is a Banach space. The equivalences observed in this section motivate the study of differential equation in Banach space which have taken attention the researchers from the last century.

3.3 Nonlinear Differential Equations in Abstract Spaces

Taking as motivation the observations in the previous section most of our efforts in this thesis are concentrated in the study of the classical Cauchy problem (CP)

$$x' = f(t, x), \qquad x(0) = x_0, \tag{3.8}$$

where $f : [0, a] \times E \to E$, E is a Banach space.
We begin by giving some counterexamples to show that the classical Peano's existence theorem of differential equations in $E = \mathbb{R}^n$, as well as the "continuation of solutions" theorem, cannot in general be extended to infinite-dimensional Banach spaces. We then offer a set of quite general sufficient conditions, more general than the monotonicity condition, that guarantee the existence and uniqueness of solutions of nonlinear abstract Cauchy problems.

It is well known that if E is finite dimensional and f is continuous in a neighborhood of $[0, a] \times E$, then by Peano existence theorem, there exists a $C^1([0, \alpha])$-curve $x : [0, \alpha] \to E$,

$\alpha < a$, (i.e., $\lim_{h\to 0} h^{-1}(x(t+h) - x(t))$ exists in the norm of E in $t > 0$, the limit being one-sided for ($t = 0$), such that $x(0) = 0$ and $x'(t) = f(t, x(t)), t \in [0, \alpha]$. The restriction $dim(E) < \infty$ seems to be necessary to impose here, the reason for believing this is that Peano's existence theorem is not applicable when $dimE = \infty$. The lack of local compactness in infinitely dimensional Banach spaces is the major problem in the subject. J. Dieudonne [32] gave a very simple example where $E = c_0$ (the space of all real-valued sequences $x = (x_n)$ with $x_n \to 0$, $\|x\|_{c_0} = \sup_n |x_n|$).

Example 3.3.1 *[32]. Let us consider the Cauchy problem which its n^{th} coordinate is given by*

$$x'_n = f_n(x) = |x_n|^{\frac{1}{2}} + \frac{1}{n}, \quad x_n(0) = 0, \tag{3.9}$$

$(x_n) \in c_0$. *Clearly, $f(x)$ satisfies Peano's existence theorem on the infinite dimension Banach space c_0, however, the Cauchy problem (3.8) has no solution in c_0. Indeed, the n^{th} coordinate of the solution of (3.9) is increasing more rapidly than the maximal solution of the scalar Cauchy problem*

$$u' = |u|^{\frac{1}{2}}, \quad u(t_0) = x_n(t_0),$$

since, for any $\varepsilon > 0$, $\lim_{n\to\infty} x_n(\varepsilon) = \frac{\varepsilon^2}{4}$. This shows that, while, $x(0) = \lim_{n\to\infty} x_n(0) \in C_0$, $x(\varepsilon)$ does not belong to c_0.

The example, just given above, seems to depend strongly on the properties of (c_0) (which is not reflexive). As far as we know, this motivates Yorke [93] to show, by means of example in Hilbert space (A Hilbert space is always reflexive), that continuity alone, of the function f, is not sufficient to prove a local existence theorem in the case where E is infinite dimensional. We state it briefly as follows.

Example 3.3.2 *[93]. Let H be the Hilbert space of all real sequences $y = (y_1, y_2, ...)$ such that $\|y\|^2 = \sum y_i^2$. Let P_n be the projections given by $P_n(y) = (0, ..., 0, y_{n+1}, y_{n+2}, ...)$, $n = 1, 2, ...$ and $P_0(y) = y$. For every $t \in \mathbb{R}$, $y \in H$ define*

$$P(t)y = \begin{cases} 0 & \textit{if } t \leq 0, \\ y & \textit{if } t \geq 1, \\ (2-2^n t)P_n y + (2^n t - 1)P_{n-1}y & \textit{if } t \in [2^{-n}, 2^{-n+1}],\ n = 1, 2, ... \end{cases}$$

The function P is continuous on $\mathbb{R} \times E$. Let

$$G(y) = y\|y\|^{-1/2} \ for\ y \neq 0\ and\ G(0) = 0.$$

Define

$$A(y) = (|y_1|, |y_2|, ...),$$

let

$$\nu = (2^{-1}, 2^{-2}, 2^{-3}, ...),$$

and,

$$F(t, y) = G(P(t)A(y)) + P(t/2)\,\nu\, max\{0, 4^{-1}t^2 - \|y\|\}.$$

Note that $F : \mathbb{R} \times E \to E$ is continuous. Yorke showed that there is no solution to the following Cauchy problem

$$x'(t) = F(t, x(t)), \quad x(0) = 0.$$

Remark 3.3.1 . *The reason that the classical proof of Peano's theorem fails in an infinite-dimensional Banach space is that the closed unit ball in a Banach space is not necessarily compact.*

Many celebrated works have been developed since the 1970s in order to obtain suitable extensions of Peano's theorem on assuming particular topologies on E. Basically two branches were born on this journey: uniformly continuity and continuity in the weak topology. The former came from the observation that, under strong topology assumptions, the function $f(t, x)$ defined in $\overline{D} = \{(t, x) \in \mathbb{R} \times E | |t - t_0| \leq a,\ |x - x_0| \leq b\}$, $dim(E) < \infty$, is automatically uniformly continuous, due to the compactness of $\overline{D}$. However, if $dim(E) = \infty$, under the additional hypothesis that the function $f(t, x)$ is (locally) Lipschizian, we can prove the (local) existence and uniqueness of solutions of (3.8).

A long of this way, we also cite Corduneanu and Krasnoselskii [26, 55] who have separately extended peano's theorem to infinite dimensional spaces in assuming additional (continuously Lipschitz) conditions.
In 1971 Chow and Schuur [22] found that, by replacing strong continuity with weak continuity and assuming the range of f to be bounded, existence result may be obtained. Also in 1971, Szep [89] provides another extension to Peano's theorem to reflexive Banach space endowed with weak topology and based his main tool to the fact that the closed set in a weak topology is weak compact. The next step was given in 1976 by kato [53]. In that paper kato observed that if the function $f(t,x) : \{[t_0, t_0+a] \times B_b\} \to E$, $B_b = \{x \in E ||x - x_0|| \leq b\}$, is weakly continuous, then all we need in order to assure the existence of solutions to the related (3.8) is the relatively weak compactness of $f([t_0, t_0+a] \times B_b)$.
We conclude that weak topology appeared as an attempt to grapple with the lack of local compactness in infinite dimensional Banach spaces. If the Banach space E is reflexive, we recover locally compactness by endowing it with the weak topology. We observe that the weak topology coincides with the strong topology in a Banach space E if, and only if, $dim(E) < \infty$.

An attempt at generalizing those previous results for non-reflexive Banach spaces was the so called measure of weak non-compactness. Probably the first work in this direction is due to E. Cramer, V. Lakshmikantham and A.R. Mitchell [27]. Roughly speaking, the idea behind this technic is to impose some condition on f involving the measure of weak non-compactness in order to, somehow, recover the local compactness lost by the fact that the Banach space we are working on is no longer reflexive. Since [27], many researchers have improved and generalized results involving assumptions on the measure of weak non-compactness. Some of the recent progress in this direction are [19, 23, 25, 41]. The only disadvantage of this theory is that, when E is not reflexive, it is difficult or even impossible to check the measure of weak non-compactness assumptions.

E. Teixeira [91] explored another line of generalization to the theory of differential equations in Banach spaces. The idea of this paper is based on the study of differential equations in locally convex spaces. Since the theory of differential equations in general locally convex spaces differs from the theory in Banach spaces, this way is out of our interest.

The remaining of this section is arranged in the following way: We first shall present a set of quite general conditions on f which guarantee that the Cauchy problem (3.8) is well posed, that is, solutions of (3.8) exist, are unique, and depend continuously on their initial data. We provide general elementary theory of nonlinear differential equations in abstract spaces. Finally, we treat differential equations in Banach spaces in two separated subsections.

Theorem 3.3.1 *[59]. Let the function $f(t,x)$ be defined and continuous in $\overline{D} = \{(t,x) \in [t_0, t_0 + a] \times B_b\}$, $B_b = \{x \in E \| |x - x_0|| \leq b\}$, $||f(t,x)|| \leq M$ for all $(t,x) \in \overline{D}$. Assume further that there exists a continuous scalar functional $V : [t_0, t_0 + a] \times B_b \times B_b \to \mathbb{R}_+$ such that*
(i) $V(t,x,y) > 0$, if $x \neq y$, $V(t,x,y) = 0$, if $x = y$.
(ii) If $\lim_{n\to\infty} V(t, x_n, y_n) = 0$ for all $t \in [t_0, t_0 + a]$, then $\lim_{n\to\infty}(x_n - y_n) = 0$.
(iii) $V(t,x,y)$ is continuously differentiable and

$$\frac{\partial V(t,x,y)}{\partial t} + \frac{\partial V(t,x,.)}{\partial x} f(t,x) + \frac{\partial V(t,.,y)}{\partial y} f(t,y) \leq 0.$$

(iv) $\frac{\partial V(t,x,y)}{\partial t}$, $\frac{\partial V(t,x,.)}{\partial x} u$ and $\frac{\partial V(t,.,y)}{\partial y} u$ are continuous in (x,y), $u \in E$ with $||u|| \leq K$, for $K > 0$. Then (3.8) possesses a unique solution on $[t_0, t_0 + \alpha]$, $\alpha M < b$. Moreover, the solution depends continuously on (t_0, x_0).

An interesting special case of the above Theorem in a Hilbert space is the following.

Theorem 3.3.2 *. Let H be a Hilbert space and let the function $f(t,x)$ be defined and continuous in $\overline{D} = \{(t,x) \in [t_0, t_0 + a] \times B_b\}$, $B_b = \{x \in H \| |x - x_0|| \leq b\}$, $||f(t,x)|| \leq M$ for all $(t,x) \in \overline{D}$. Furthermore, assume that $-f$ is a monotonic function, that is, there exists a constant K such that*

$$Re[f(t,x) - f(t,y), x - y] \leq K||x - y||^2, \;\; t \in [t_0, t_0 + \alpha], \;\; x, y \in H.$$

Then the conclusion of the above theorem is satisfied.

Theorem 3.3.3 . *Let the function $f(t,x)$ be defined in $\overline{D}$, $\overline{D} = \{(t,x) \in \mathbb{R} \times E \,:\, |t-t_0| \leq a,\ |x-x_0| \leq b\}$, $||f(t,x)|| \leq M$ for all $(t,x) \in \overline{D}$. Assume further that $||f(t,x) - f(t,\overline{x})|| \leq L|x-\overline{x}|$, for all $(t,x),(t,\overline{x}) \in \overline{D}$, $aM < b$. Then there exists one and only one (strongly) continuously differentiable solution of (3.8) on $|t-t_0| \leq a$.*

Remark 3.3.2 . *If E is a Banach algebra over a field F, the function $f(t,x) = ax$ for $a, x \in E$ with fixed a satisfies the hypotheses of the above theorem, therefore the Cauchy problem:*

$$x' = ax, \qquad x(0) = e,$$

has a unique solution. This solution is given by

$$x(t) = e + \sum_{i=1}^{\infty} \frac{t^n\, a^n}{n!}.$$

The above formula has already been taken as the definition of the exponential function $exp(at)$ in E.

3.3.1 Cauchy Problem in Reflexive Banach Space

In this subsection we provide the most general extensions of Peano's theorem in reflexive Banach space. The study for weak solutions of the Cauchy differential equation in reflexive Banach spaces was initiated by, among others, Szep [89]. However, we begin with Chow and Schuur's criterion for the existence of solution of (3.8).

Definition 3.3.1 *[22]. A solution of (3.8) on some interval I is a function $x(t)$ defined on I, weakly continuous (i.e., for every $F\ inE^*$, $F(x(t))$ is a continuous real-valued function), and for almost every $t \in I$, $x'(t) = f(t,x(t))$ (where $x'(t)$ is understood to be the weak limit of the quotient $\frac{x(t+h)-x(t)}{h}$ as $h \to 0$).*

Definition 3.3.2 *[22]. A function $f : [t_0, t_0+a] \times E \to E$ is said to be quasi-weakly continuous at (t_0, x_0) if it satisfies*

$$f(t_0, x_0) = \cap_{N=1}^{\infty} cl\ co\ f_N(t_0, x_0), \tag{3.10}$$

where $f_N(t_0, x_0) = \cup\{f(t,x) : |t - t_0| < \frac{1}{N}, |F_i(x - x_0)| < \frac{1}{N}, i = 1, \ldots.., N\}$ and, $cl\ co\ f_N(t_0, x_0)$ denotes the closed convex hull of $f_N(t_0, x_0)$. We notice that if E is finite dimensional, condition (3.10) is equivalent to saying that f is continuous, moreover if $f : [t_0, t_0 + a] \times E_w \to E_w$ is continuous (i.e., E is endowed with the weak topology), then f satisfies (3.10). Thus, condition (3.10) is a reasonable generalization of continuity.

Theorem 3.3.4 *[22]. Let $f : [t_0, t_0 + a] \times E_w \to E_w$ satisfy (3.10). Then $x(t)$ defined on some interval $I \in [t_0, t_0 + a]$ is a solution of (3.8) in the sense of the just above definition if and only if for every $N > 1$, $t \in I$ there exists $\eta > 0$ such that*

$$0 < h < \eta \Rightarrow \frac{x(t+h) - x(t)}{h} \in cl\ co\ f_N(t_0, x_0).$$

Theorem 3.3.5 *[22]. Assume that in a neighborhood of $(t_0, x_0) \in [t_0 - a, t_0 + a] \times E$, $f : [t_0 - a, t_0 + a] \times E \to E$ is bounded in a norm and satisfies (3.10).*
Then there exists a solution $x(t)$ of the Cauchy problem (3.8) on some interval $I \subset [t_0 - a, t_0 + a]$ containing t_0. Further $x(t)$ is absolutely continuous and satisfies $x'(t) = f(t, x(t))$ a.e. on I (where $x'(t)$ is understood as the strong limit of $\frac{x(t+h)-x(t)}{h}$ as $h \to 0$).

Definition 3.3.3 *[63]. A function $f(t,x)$ is said to be weakly-weakly continuous at (s,y) if given $\varepsilon > 0$ and $\varphi \in E^*$ there exist $\delta(\varepsilon, \varphi) > 0$ and $\daleth(\varepsilon, \varphi)$ a weakly open set containing y such that $|\varphi(f(t,x) - f(s,y))| < \varepsilon$ whenever $|t - s| < \delta$ and $x \in \daleth$.*

Theorem 3.3.6 *[89]. Let E be a reflexive Banach space and f be a weakly-weakly continuous function on $A : 0 \le t \le a, \|x\| \le b$. Let $\|f(t,x)\| \le M$ on A. Then (3.8) has at least one weak solution defined on $[0, \alpha]$, $\alpha = min\{a, \frac{b}{M}\}$.*

In 1976 kato [53] proved the existence of both local and global existence of strongly continuous, once weakly continuously differentiable solutions of the Cauchy problem (3.8).

Theorem 3.3.7 *[53]. Let f be a weakly continuous mapping from $[0, a] \times S(x_0, r)$ into E. Suppose further that the range $f([0, a] \times S(x_0, r))$ is relatively compact in E_w . Then the Cauchy problem (3.8) has at least one solution x defined on some interval $[0, \alpha]$.*

If E is a reflexive Banach space then we have the following result.

Theorem 3.3.8 *[53]. Let E be a reflexive Banach space and let $f : [0, \infty) \times E \to E$ be a weakly continuous function. Then for each $r > 0$ there exist $\alpha(r) > 0$ such that for each x_0 in E with $||x_0|| \leq r$, the Cauchy problem (3.8) has at least one solution defined on $[0, \alpha(r)]$.*

It turned out then that weak topology is the right assumption on the reflexive Banach space E to prove existence of weak solution of the Cauchy problem (3.8). However, if E is nonreflexive Banach space the situation is quite different. Astala in [6], proved that a Banach space E is reflexive if and only if the (3.8) admits a local solution for every weakly continuous field. Thus there is no hope to extend Peanos theorem in the weak topology setting to non-reflexive spaces. A similar study of the Cauchy problem (3.8) in nonreflexive Banach spaces is given in the next subsection. The existence and uniqueness of global solutions for a class of abstract Cauchy problem section will be considered as well.

3.3.2 Cauchy Problem in Nonreflexive Banach Space

This subsection is devoted to give a brief study the Cauchy problem (3.8) in arbitrary nonreflexive Banach spaces. In their pioneer work Cramer, Lakshmikantham and Mitchell [27] proved the existence of weak solution of differential equation in nonreflexive Banach space imposing weak compactness type condition. For this purpose they introduced the notion of measure of noncompactness which we give herein. They also imposed weak dissipative type conditions and prove an existence and uniqueness theorem.

There are many results depend on using measure of weak noncompactness for imposing weak compactness type conditions will be given herein as well. Kuratowski [58] defines $\omega(A)$, a measure of the noncompactness of A. He defines $\omega(A)$ to be the infimum of all $\epsilon > 0$, for which there exists a finite covering of A by sets of diameter ϵ. Many researchers have used the condition that f is ω-Lipschitzean, that is that there is an $L > 0$

such that

$$\omega(f(A)) \leq L\omega(A),$$

for bounded A. We now list some necessary definition and properties for the measure of weak noncompactness which is a generalization of that of Kuratowski.

Definition 3.3.4 *.For a bounded subset A of a Banach space E, the measure of weak noncompactness $\omega(A)$ is a real valued function defined by*
$\omega(A) = inf\{\varepsilon > 0 : \ for\ each\ \Gamma \in E^*, \|\Gamma\| = 1\ there\ exist\ x_1, x_2, ..., x_{n_\Gamma}\ such\ that\ A \subset \cup_{i=1}^{n}\{x : |\Gamma(x - x_i)| \leq \varepsilon\}\}$.

This measure has some properties in the following lemma.

Lemma 3.3.1 *. Let A and B are bounded subsets of E and $\{a_n\}$ and $\{b_n\}$ be bounded sequences in E. Then:*
1) if $A \subset B$ then $\omega(A) \leq \omega(B)$.
2) if E is weakly complete then $\omega(A) = 0$ if and only if A is relatively weakly compact.
3) $\omega(A) = \omega(\overline{A})$ where $\overline{A}$ denotes the weak closure of A.
4) $\omega(tA) = t\omega(A)$ for $t \geq 0$.
5) $\omega(A) = \omega(CoA)$. where CoA denote the convex of a set A.
6) $\omega(A \cup B) = max\{\omega(A), \omega(B)\}$
7) $\omega(x + A) = \omega(A)$ for all $x \in E$
8) $\omega(A + B) \leq \omega(A) + \omega(B)$
9) $\omega(\{a_n\}) - \omega(\{b_n\}) \leq \omega(\{a_n - b_n\})$
10) given $\varepsilon > 0$ if for each $\Gamma \in E^*$ with $\|\Gamma\| = 1$ there exist $M = (\Gamma, \varepsilon)$ such that $n > M$ implies $|\Gamma(x_n)| < \varepsilon$ then $\omega(x_n) \leq \varepsilon$.
11) if $\{a_n\}$ is a sequence of nonempty weakly closed and bounded subset of E such that $a_1 \supset a_2 \supset ..... \supset a_n \supset .....$ and $lim_{n\to\infty}\omega(a_n) = 0$ then $\cap_{n=1}^{\infty} a_n \neq \phi$.

It is will known that $\omega(A) = 0$ when E is reflexive since every bounded set in a reflexive Banach space is relatively weakly compact.

Now we follow [27] to impose weak compactness type condition on employing the measure of noncompactness discussed above.

Theorem 3.3.9 *[27]. Let E_w be weakly complete and let f be a weakly-weakly continuous function on $D : \{t_0 \leq t \leq t_0 + a, \|x\| \leq b\}$, $\|f(t,x)\| \leq M$ on D. Assume further that*

$$\omega(f(I \times A)) \leq \ g(\omega(A))$$

where $I = [t_0, t_0 + \alpha]$, $A \subset \{\ x \in E : \|x\| \leq b\}$ and $g \in C[\mathbb{R}^+, \mathbb{R}^+]$. Assume that $u(t) = 0$ is the unique solution of the auxiliary Cauchy problem $u' = g(u), u(t_0) = 0$ on $[0, \alpha]$.
Then there exists a solution $x(t)$ for the Cauchy problem (3.8) on $[0, \alpha]$, where $\alpha = min(a, \frac{b}{M})$.

In 1980 Boudourides [12] proved a local existence of solution by assuming f to be a weakly continuous and ω-Lipschitzian where ω is a measure of weak noncompactness in the weak topology (as introduced by De Blasi [28]). Of course E is still a nonreflexive Banach space.

Definition 3.3.5 *[28]. For a bounded subset A of E, the measure of weak noncompactness $\omega(A)$ is a real valued function defined by*
$\omega(A) = inf\{\varepsilon > 0 : A \subset U + tB_0, U \subset E \ weakly \ \ compact\}$
where B_0 denote the closed unit ball of E.

The properties of the measure ω as introduced by Blasi are analogous to that of measure of weak noncompactness and then are omitted.

Definition 3.3.6 *[12]. By a solution we mean a strongly continuous, once weakly differentiable function $x : [t_0, t_0 + \alpha] \to E$ satisfying (3.8) in $[t_0, t_0 + \alpha]$, with x' denoting the weak derivative.*

Theorem 3.3.10 *[12]. Let f be a weakly continuous function on $I \times D : I = \{t_0 \leq t \leq t_0 + a\}, D = \{x \in E : \|x\| \leq b\}$, $\|f(t,x)\| \leq M$ on $I \times D$. Assume further that f is ω-Lipschitzian i.e, there exist a $L \geq 0$ such that*

$$\omega(f(I \times A) \leq L\omega(A), \quad A \subset D$$

Then the Cauchy problem (3.8) has a solution on $[0, \alpha]$, where $\alpha = min(a, \frac{b}{M})$ and $\alpha L < 1$.

Remark 3.3.3 . *If f is weakly continuous and, $\overline{f(I \times D)}$ is weakly compact (i.e., the closed set in a weak topology is weakly compact), then obviously f is ω-Lipschitzian. Furthermore, if f is weakly continuous and E is a reflexive Banach space, f is trivially ω-Lipschitzian. Thus the results of Kato and Szep [53, 89] are special cases of the above theorem. We state them as separate corollaries.*

Corollary 3.3.1 *If f is weakly continuous and, $\overline{f(I \times D)}$ is weakly compact. Then the Cauchy problem (3.8) has a solution on $[0, \alpha]$, where $\alpha = min(a, \frac{b}{M})$, and $\|f(t, x)\| \leq M$ on $I \times D$.*

Corollary 3.3.2 . *If E is a reflexive Banach space, f is weakly continuous and (strongly) bounded $\|f(t, x)\| \leq M$ on $I \times D$. Then the Cauchy problem (3.8) has a solution on $[0, \alpha]$, where $\alpha = min(a, \frac{b}{M})$.*

Afterwards, Kubiczyk and Szufla [57] proved, under conditions similar to that imposed by [27], that the set of all weak solutions defined on $[0, 1]$ is nonempty, compact and connected in $C_w([0, 1], E)$ (the set of all weak continuous function from $[0, 1]$ into a Banach space E endowed with the topology of uniform weak convergence). Since Kubiczyk and Szufla conditions are given in terms of the kamke function, we then give it herein.

Definition 3.3.7 . *A nondecreasing function $g(t, u) : I \times \mathbb{R}^+ \to \mathbb{R}^+$ is called kamke function if it satisfies the following conditions*
(i) $g(t, u)$ is a Caratheodry function, i.e. measurable with respect to t and continuous with respect to u.
(ii) there exist an integrable function $M(t)$ such that $g(t, u) \leq M(t)$.
(iii) for all $t \in I;\ g(t, 0) = 0$.
(iv) $u(t) \equiv 0$ is the only absolutely continuous function which satisfies $u' \leq g(t, u(t))$ a.e. on I and such that $u(0) = 0$.

The condition (iv) can be replaced by the following condition.
(iv') $u(t) \equiv 0$ is the unique solution of the integral equation $u(t) \leq \int_0^t g(t, u(s))ds$ on I with $u(0) = 0$.

Theorem 3.3.11 *[57]. Let f be a weakly-weakly continuous function on $I \times D : I = \{t_0 \leq t \leq t_0 + a\}, D = \{x \in E : \|x\| \leq b\}$, $\|f(t,x)\| \leq M$ on $I \times D$. Moreover, assume that E_w-the space E endowed with weak topology, is sequentially weak complete we have*

$$\omega(f(I \times A)) \leq \; g(\omega(A)), \quad A \subset D$$

where g is Kamke function which is independent of t. Then the set of all weak solutions of the Cauchy problem (3.8) defined on $[0, \alpha]$ is nonempty, compact and connect in $C_w([0, \alpha], E)$

In a nonreflexive Banach space, Using a compactness hypothesis involving a weak measure of noncompactness, Papageorgiou [77] proved an existence result that generalizes earlier theorems by Chow-Shur, Szep, Kato and Cramer-Lakshmikantham-Mitchell. It should be noted that the weak measure of noncompactness used in the following theorem is that introduced by De Blasi [28].

Definition 3.3.8 . *By a solution of (3.8) we mean a strongly continuous, once weakly differentiable function satisfying (3.8) on $[0, \alpha]$, with x' denoting the weak derivative.*

Theorem 3.3.12 *[77]. Let $f : [0, a] \times E \to E$ be such that*
(i) f is weakly continuous and $\|f(t,x)\| \leq M$ for all t, x
(ii) for every subset A of E we have

$$lim_{a \to 0} \omega(f([0, a] \times A)) \leq \; g(t, \omega(A)),$$

where $g(.,.)$ is a kamke function. Then the Cauchy problem (3.8) has at least one solution.

It should be noted that Cramer, Lakshmikantham and Mitchell theorem can be considered as a corollary of the above theorem.

Corollary 3.3.3 *[27]. Let the function $f : [0, a] \times E \to E$ be a weak continuous and $\|f(t,x)\| \leq M$ for all t, x. Assume further that, for all bounded set A of E,*

$$\omega(f(I \times A)) \leq \; g(\omega(A)),$$

where g is Kamke function. Then (3.8) admits a solution in $t \in [0, \alpha]$.

Also we notice that, if E is reflexive, by compactness, every bounded set is relatively weakly compact, and hence szep and Chow-Shur results are deduced.

Corollary 3.3.4 *[89]. Let the function $f : [0, a] \times E \to E$ be a weak continuous and $\|f(t, x)\| \leq M$ for all t, x. Then (3.8) admits a solution in $t \in [0, \alpha]$.*

Also, if E is any Banach space and f is a compact function, then Kato result is deduced.

Corollary 3.3.5 *[53]. Let the function $f : [0, a] \times E \to E$ be a weak continuous and $\|f(t, x)\| \leq M$ for all t, x. Assume further that $f(t, x)$ is a compact function. Then (3.8) admits a solution in $t \in [0, \alpha]$.*

Counterexamples given in c_0 by Dieudonne [32] as well as that given in l_2 by Yorke [93] show that, in infinite-dimensional spaces, Peano's existence theorem need not necessarily be true. A natural question then is that of asking whether there could exist infinite-dimensional Banach spaces on which Peano's theorem holds, or otherwise, whether the truth of Peano's theorem is a characterization of the finite dimensionality. Cellina in [21] offers only a partial answer to this question. He proved that there exist no nonreflexive spaces on which Peano's theorem holds.

Theorem 3.3.13 *[21]. If E is nonreflexive Banach space. Then there exists a continuous function $f : [0, a] \times E \to E$ such that the Cauchy problem (3.8) ($x(0) = 0$) admits no solution on any nonempty interval I containing 0.*

Now we consider the Cauchy problem for an ordinary differential equation in E.

$$x'(t) = f(x(t)), \quad x(0) = 0. \tag{3.11}$$

When $E = \mathbb{R}^n$, f is assumed to be Lipschitz continuous, classical theorem states the existence and uniqueness of a solution (3.11). In a nonreflexive Banach space E O'regan [76] proved, by using a fixed point theorem, existence of weak solutions of the Cauchy problem(3.11).

Theorem 3.3.14 *[76]. Let Υ be a nonempty, bounded, convex, closed set in a Banach space E. Assume that $f : \Upsilon \to \Upsilon$ is weakly sequentially continuous and α ω-contractive (here $0 \leq \alpha < 1$), ω denotes the Blasi measure of weak noncompactness. Then f has at least one fixed point in Υ.*

Theorem 3.3.15 *[76]. Let E be a Banach space, Υ a nonempty, bounded, closed, convex, equicontinuous subset of $C([0,a],E)$ and let $f : \Upsilon \to \Upsilon$ be weakly sequentially continuous (i.e., for any sequence x_n in Υ with $x_n(t) \to x(t)$ in E_W for each $t \in [0,a]$, then $f(x_n(t)) \to f(x(t))$ in E_W for $t \in [0,a]$). Assume further that there exists $0 \leq \alpha < 1$ with $\omega(f(A)) \leq \alpha\omega(A)$ for all subsets $A \subset \Upsilon$. Then the Cauchy problem(3.11) has a solution in Υ.*

Theorem 3.3.16 *[76]. Let E be a Banach space, Υ a nonempty, bounded, closed, convex, equicontinuous subset of $C([0,a],E)$ and let $f : \Upsilon \to \Upsilon$ be weakly sequentially continuous. Assume further that $f(\Upsilon(t))$ is weakly relatively compact in E for each $t \in [0,a]$. Then the Cauchy problem (3.11) has a solution in Υ.*

We now discuss a special case of the Cauchy problem (3.11), namely,

$$x(t) = x_0 + \int f(t, x(s)ds, \tag{3.12}$$

where the integral here is understood to be the Pettis integral.

Theorem 3.3.17 *[76]. Assume that $(C1) - (C3)$ below hold:*
$C1)$ for each $t \in [0,a]$, $f_t = f(t,.)$ is weakly sequentially continuous,
$C2)$ for each continuous $x : [0,a] \to E$, $f(.,x(.))$ is Pettis integrable on $[0,a]$,
$C3)$ for any $\beta > 0$, there exists $M_\beta \in L^1[0,a]$ with $||f(t,x)|| \leq M_\beta$ for a.e. $t \in [0,a]$ and all $x \in E$ with $||x|| \leq \beta$.
In addition assume that $\mathcal{F}(\Upsilon(t))$ is weakly relatively compact in E for each $t \in [0,a]$, where

$$\mathcal{F}x(t) = x_0 + \int f(t, x(s))\, ds.$$

Then (3.12) has a solution in $C([0,a],E)$.

The proof of the above theorem is based on the fact that $\mathcal{F} : C([0,a],E) \to C([0,a],E)$ has a fixed point in $C([0,a],E)$. In this way D. O'rgan established existence of weak solutions for the Cauchy problem (3.8). This, among others, illustrates the power of his existence principle. It should also be noted that the conclusion of the O'rgan's theorem holds even if E is reflexive, in fact the condition that $f\Upsilon(t)$ is weakly relatively compact in E for each $t \in [0, a]$ is automatically satisfied since a subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology, ($f : \Upsilon \to \Upsilon$ and Υ is bounded in $C([0, a], E)$).

3.4 Uniqueness Results for a Class of Abstract Cauchy Problem

For the sake of completeness we provide various basic uniqueness criterion for the Cauchy problem (3.8) when E is an arbitrary Banach space. Goldstein [40], using dissipative type condition, gave an extension of Nagumo's condition as follows.

Theorem 3.4.1 *[40]. Let f have a domain D contained in $[0, c) \times E$ and range contained in E. Suppose that there is a duality map J of E such that*

$$< f(t, x) - f(t, \overline{x}), J(x - \overline{x}) >\leq t^{-1}\|x - \overline{x}\|^2,$$

whenever $t > 0$, $(t, x), (t, \overline{x}) \in D$. Then there exist at most one solution x of the Cauchy problem (3.8) defined on $[0, c)$.

Goldstein [40] defined the solution of (3.8) in the following sense.
(i) x is strongly absolutely continuous and has a strong right derivative x' a.e. in $[0, c)$,
(ii) $x'(t) = f(t, x(t))$ a.e. in $[0, c)$,
(iii) $x(0) = x_0$,
(iv) strong $lim_{t\to 0^+} t^{-1}(x(t) - u(0)) = x_1$.

Remark 3.4.1 . *When E is a (real or complex) Hilbert space, $< .,. >$ is the inner product* *on E and the duality mapping J is the identity operator; thus the above result generalizes the corresponding results in [29, 69].*

A generalization of Osgood type condition in Banach spaces is given by Goldstein [40] as follows:

$$< f(t,x) - f(t,\overline{x}), J(x-\overline{x}) >\leq \|x-\overline{x}\| g(t, \|x-\overline{x}\|),$$

whenever $t > 0$, $(t,x),(t,\overline{x}) \in D$, g is nonnegative measurable function defined on $(0,c)\times[0,\infty)$ such that $u=0$ is the only solution of $u(t) \leq \int_0^t g(s,u(s))ds$ a.e. in $[0,c)$.
kato [53] proved the global existence and uniqueness of solutions to (3.8). A functional $< .,. >: E\times E \to \mathbb{R}$ is defined by:

$$< x, y >= \frac{1}{2} lim_{h\to+0} \frac{1}{h}(\|x+hy\| - \|x-hy\|).$$

Theorem 3.4.2 *[53]. Let E be a reflexive Banach space and $f : [0,\infty)\times E \to E$ be a weakly continuous function. Assume further that*

$$< x-y, f(t,x)-f(t,y) >\leq \beta(t)\|x-y\|, \tag{3.13}$$

for all $x,y \in E$ and a.e.$t \in (0,\infty)$,where $\beta \in L^1_{loc}(0,\infty)$ be a locally integrable function on $(0,\infty)$. Then for each x_0 in E the Cauchy problem (3.8) has a unique solution $x(t)$ defined on $[0,\infty)$.

If E is a Hilbert space with inner product denoted by $(.,.)$. Then it is easy to see that $< x, y >= \frac{Re(x,y)}{||x||}$, for $x \neq 0$ and$y \in E$.
Hence, the condition (3.13) becomes

$$Re(f(t,x)-f(t,y), x-y) \leq \beta(t)\|x-y\|^2,$$

for all $x,y \in E$ and a.e.$t > 0$. This implies that $f(t,.) - \beta(t)I$ is dissipative for a.e.$t > 0$ where I denotes the identity mapping.

Another result of Cramer, Lakshmikantham and Mitchell in their paper [27] is to utilize a weak dissipative type condition to prove existence and uniqueness of solutions of the Cauchy problem (3.8).

Definition 3.4.1 *[27]. A function $f(t,x)$ is said to be weakly-weakly uniformly continuous if given $\varepsilon > 0$ and $\varphi \in E^*$ there exist $\delta(\varepsilon,\varphi) > 0$ and $\{\varphi_i | \varphi_i \in E^*,\ i = 1,2,...,n_{(\varepsilon,\varphi)}\}$ such that $|\varphi(f(t,x) - f(s,y))| < \varepsilon$ whenever $|t-s| < \delta$ and $|\varphi_i(x-y)| < \varepsilon$, $i = 1,2,...,n_{(\varepsilon,\varphi)}$.*

Theorem 3.4.3 *[27]. Let the conditions $(C1) - (C3)$ be fulfilled.*

$C1)$ If E_w is weakly complete, f is a weakly-weakly uniformly continuous function on $D : \{t_0 \le t \le t_0 + a, \|x\| \le b\}$ and $\|f(t,x)\| \le M$ on D.

$C2)$ For each $\varphi \in E^*$ with $||\varphi|| = 1$,

$$\overline{\lim_{h \to 0^+}} \frac{|\varphi(x - y + h(f(t,x) - f(t,y)))| - |\varphi(x-y)|}{h} \le g(t, \varphi(x-y)),$$

where $g \in C([t_0, t_0 + a] \times [0, 2b], \mathbb{R}^+)$.

$C3)$ $u(t) = 0$ is the only solution of the auxiliary Cauchy problem $u' = g(t,u),\ u(t_0) = 0$.

Then there exists a unique solution $x(t)$ for the Cauchy problem (3.8) on $[0,\alpha]$, where $\alpha = min(a, \frac{b}{M})$.

Number of known results are extended by Arrate[4]. In fact, M.Arrate provides a uniqueness criteria which contains those of Goldstein as a particular cases and extends,by using dissipative type condition, to Banach spaces Gard's and Roger's results [37, 83] as the following theorem shows.

Theorem 3.4.4 *Let $f : (0,a) \times E \to E$ and let g be a function continuous in $[0,a)$, differentiable in $(0,a)$, and such that $g(0) = 0$, $g(t) > 0$ for all $t > 0$. Assume further that x, $\bar{x}$ are two solutions of the Cauchy problem (3.8) in $[0,\alpha)$, $\alpha < a$ verifying*
$\|f(t,x) - f(t,\bar{x})\| = O[g'(t)]$. Then, if $f(t,.) - \frac{g'(t)I}{g(t)}$ is dissipative for all $t \in (0,\alpha)$ it follows that $x(t) \equiv \bar{x}(t)$.

Afterwards, Arrate and Garcia [5] extended Arrate, Gard, Goldstein, kamke and Rogers's results [4, 37, 40, 50, 83] using the inner semi-product defined as follows where $k(x)$ is a duality mapping of E is a function.

Definition 3.4.2 *[5]. For $x, \bar{x} \in E$ the inner semi-products of $x, \bar{x}$ defined by*

$$\langle x, \bar{x} \rangle_- = inf\{Re\langle x, k(x)\rangle : k \in K(x)\}$$

$$\langle x, \bar{x}\rangle_+ = sup\{Re\langle x, k(x)\rangle : k \in K(x)\}.$$

The main theorem in [5] concern with $f : (0, a) \times B(x_0, b) \to E$ and $g(t)$ defined as in the previous theorem with $g'(t) \neq 0$. Further, let $G(t, u)$ be a Kamke function, where, $u \equiv 0$ is the unique solution of $u' = G(t, u)$ such that

$$lim_{t\to 0^+} \frac{u(t)}{g(t)} = 0$$

Theorem 3.4.5 *[5]. Let $f : (0, a) \times B(x_0, b) \to E$ and let $x(t), \bar{x}(t)$ be two solutions of the Cauchy problem (3.8) defined on $[0, \alpha), \alpha < a$ such that*
(i)$\|f(t, x(t)) - f(t, \bar{x}(t)\| = O[g'(t)]$ as $t \to 0^+$,
(ii)$\langle f(t, x_1) - f(t, x_2), x - x_2\rangle_- \leq G(t, \|x_1 - x_2\|)\|x_1 - x_2\|$,
for all $(t, x_1), (t, x_2) \in (0, a) \times B(x_0, b)$.
Then $x(t) \equiv \bar{x}(t)$ on $[0, \alpha)$.

All the above uniqueness criteria depend on assuming dissipative type conditions. This type of conditions seems to be unnecessary to impose as we shall show in the next chapter.

Chapter 4

Abstract Versions of Majorana's Uniqueness Criterion

The results obtained in [65] crucially depend on analyzing the auxiliary (not differential) equation

$$u = tf(t, u), \tag{4.1}$$

where u is the unknown and t a parameter whose role will be specified in the sequel and, establishing a very close relation between (4.1) and Cauchy problem

$$x' = f(t, x), \qquad x(0) = 0, \tag{4.2}$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$. The privileges of Majorana's uniqueness criterion [65], as mentioned above, enforce us to ask whether or not Majorana's results can be extended to an arbitrary abstract space?. The answer to the above question is unfortunately negative. Really, Majorana's results are not directly extendable to an arbitrary abstract space as the following example shows.

Example 4.0.1 . *Let us consider the Cauchy problem:*

$$x' = f(t, x), \quad x(0) = 0, \tag{4.3}$$

where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$f(t,x) = \begin{cases} \left(\frac{2}{\sqrt{\|x\|}}(x_1 + x_2), \frac{2}{\sqrt{\|x\|}}(x_2 - x_1)\right) & \textit{if } x \neq 0, \\ 0 & \textit{if } x = 0. \end{cases}$$

In polar coordinates (4.3) becomes

$$r' = 2\sqrt{r}, \quad \theta' = \frac{-2}{\sqrt{r}}.$$

Thus, besides the trivial solution $x = 0$, there is at least one another given by

$$x = \begin{cases} \left(t^2 \cos \ln \frac{1}{t^2}, t^2 \sin \ln \frac{1}{t^2}\right) & \textit{if } t \neq 0 \\ 0 & \textit{if } t = 0. \end{cases}$$

We conclude that (4.3) satisfies the assumptions of Theorem 2.1.2 (Chapter 2), however, one can't find more than the trivial root to the auxiliary scalar equation ASE corresponding to (4.3), namely,

$$u_1 = \lambda(u_1 + u_2), \quad u_2 = \lambda(u_2 - u_1),$$

where λ is an arbitrary scalar.

One of the ideas that appeared toward the generalization of Majorana's results for Banach spaces is to find an adequate scalar equation to retain the above mentioned relation, see [2, 3, 42]. The previous preliminaries guide us to consider separately Majorana's results in finite as well as in infinite dimensional Banach spaces. This chapter contains the abstract formulation of Majorana's results. Our main concern in the first section of this chapter is the classical Cauchy problem (4.1), where f takes values in a real or complex Banach space E with $(dimE < \infty)$. The aim of Section 4.2 is to prove a new theorem for the Cauchy problem for an ordinary differential equation in infinite dimensional Banach spaces.

4.1 Uniqueness Results for Ordinary Differential Equations in Finite Dimensional Banach Spaces

We are in a position to give our first participation in this thesis [2] in which we provide an abstract version of Majorana's uniqueness theorem in finite dimensional Banach spaces. The original version of Majorana's uniqueness theorem contained results similar to those presented here, but with scalar Cauchy problem. A way to provide a version of Majorana's uniqueness theorem in Banach space consists in replacing (4.1) with a suitable one. Our main concern in this section is the classical Cauchy problem (4.2), where f takes values in a real or complex Banach space E with ($dimE < \infty$), x and 0 are in E. The underlying idea to provide counterpart to (4.1) is based on duality mappings concept. In order to make the discussion somewhat self contained, we first introduce an elementary property of the duality map of a Banach space and of a "semi-inner product" derived from it.

Throughout the following $\|.\|$ stands for the norm in E. Let E^* be the anti-dual of E, i.e., the space of all continuous anti-linear (or conjugate linear) functionals on E. The image of $x \in E$ under $x^* \in E^*$ will be denoted by $< x, x^* >$. For each $x \in E$, $K(x)$ denotes the nonempty set of all $x^* \in E^*$ for which $< x, x^* >= \|x\|^2 = \|x^*\|^2$, where for each $x^* \in E^*$, we define

$$\|x^*\| = sup\{|x^*(x)| : x \in E, \|x\| \leq 1\}$$

A duality mapping of E is a function $k : E \to 2^{E^*}$ such that $k(x) \in K(x)$ for each $x \in B$. As mentioned in section 3.1 we shall rely on the following simple observation due to [51, 7].

Suppose that $(C1), (C2)$ below are fulfilled

(C1) $x(t)$ has a weak derivative $x'(t) \in E$,

(C2) $\|x(t)\|$ is differentiable. Then

$$\|x(t)\| \frac{d}{dt} \|x(t)\| = Re < x'(t), x_t^* >, \tag{4.4}$$

for each $x_t^* \in K(x(t))$.

By a solution of (4.2) we mean a C^1 function $x : [0,a] \to E$, $a < 1$, (i.e., $\lim_{h\to 0} h^{-1}(x(t+h)-x(t))$ exists in the norm of E in $t > 0$, the limit being one-sided for $t=0$), such that $x(0)=0$ and $x'(t) = f(t,x(t)), t \in [0,a]$.

Theorem 4.1.1 . *Let the abstract function $f(t,x)$ be continuous on $[0,1]\times E$ and satisfy $f(t,0) = 0$ for every $t \in [0,1]$. Further let (4.2) admit two different solutions defined in $[0,a]$($a<1$). Then, for every $\varepsilon > 0$, and for every duality mapping $k : E \to 2^{E^*}$, there exists $t \in [0,a]$ such that the following scalar equation*

$$Re < f(t,u), k(u) >= \frac{1}{2t}\|u\|^2, \tag{4.5}$$

has at least two different roots u with $\|u\| < \varepsilon$, where u is the unknown and t a parameter whose role will be specified in the following.

Proof. The techniques employed in this proof are similar to those of Majorana. It follows, by the assumption $f(t,0)=0$, that (4.2) has the zero solution, so we assume that (4.2) has the solution $\varphi(t) \neq 0$.

Let $\varepsilon > 0$ and $k : E \to 2^{E^*}$ be given. Since $u = 0$ is a root of (4.5) for every $t \in [0,a]$, it is sufficient to show that there exists $t \in [0,a]$ for which (4.5) is satisfied by some $u \neq 0$ with $\|u\| < \varepsilon$. Let a real-valued function $A_k(t)$ be defined by setting

$$A_k(t) = \begin{cases} \frac{\|\varphi(t)\|^2}{t} & t \neq 0 \\ 0 & t = 0, \end{cases}$$

$t \in [0,a]$. Of course $A_k(t)$ is continuous in $[0,a]$, differentiable in $(0,a)$, and for every t in $(0,a)$ and for every $k(\varphi(t)) \in K(\varphi(t))$, using (4.4), we have

$$A_k'(t) = \frac{1}{t^2}[Re < 2tf(t,\varphi(t)), k(\varphi(t)) > -\|\varphi(t)\|^2]. \tag{4.6}$$

Now, fix $t_2 \in (0,a)$ with $\varphi(t_2) \neq 0$ such that $\|\varphi(t)\| < \varepsilon$ for every $t \in [0,t_2]$. Denote $t_1 = \sup\{t \in [0,t_2] : A_k(t) = 0\}$. Clearly $\varphi(t_1) = 0$ and $\varphi(t) \neq 0$ for every $t \in (t_1, t_2]$. At this point there are just two possibilities:

P1: If there exists an $t \in (t_1, t_2]$ such that $A'_k(t) = 0$, then from (4.6) for such a t, (4.5) is satisfied by $u = \varphi(t)$. Hence the proof is accomplished just taking these t and $u = \varphi(t)$.
P2: Otherwise if $A'_k(t) \neq 0$ for every $t \in (t_1, t_2]$. According to Darboux property $A'_k(t)$ has a constant sign in $(t_1, t_2]$, then we take $u = \varphi(t_2)(\neq 0)$, $k(\varphi(t_2)) \in K(\varphi(t_2))$, and define

$$G(t) = Re < 2tf(t, \varphi(t_2)), k(\varphi(t_2)) > -\|\varphi(t_2)\|^2, \quad for\ every\ t \in [0, a].$$

Now let us suppose that $A'_k(t) > 0$ for every $t \in (t_1, t_2]$. $G(0) = -\|\varphi(t_2)\|^2 < 0$. On the other hand, we have $G(t_2) = t_2^2 A'_k(t_2) > 0$. It follows, by continuity of G, that there exists an $t \in (0, t_2]$ such that $G(t) = 0$. It remains to show that the case $A'_k(t) < 0$ is impossible. Assume that $A'_k(t) < 0$, for every $t \in (t_1, t_2]$, using (4.6),

$$t\frac{d}{dt}\|\varphi(t)\|^2 = 2Re < t\varphi'(t), k(\varphi(t)) > \leq \|\varphi(t)\|^2,$$

$$t\frac{d}{dt}\|\varphi(t)\|^2 - \|\varphi(t)\|^2 \leq 0,$$

$$\frac{d}{dt}\frac{1}{t}\|\varphi(t)\|^2 \leq 0,$$

for $t > t_1$. Moreover,

$$\lim_{t \downarrow t_1} \frac{\|\varphi(t)\|^2}{t} = 0,$$

because of continuity of φ, and $\varphi(t_1) = 0$. Thus, the continuous non-negative function $\frac{\|\varphi(t)\|^2}{t}$ is non-increasing for $t > t_1$, while its limit, as t approaches t_1 from the right, is zero. Consequently, one must have $\frac{\|\varphi(t)\|^2}{t} = 0$ for $t > t_1$, which contradicts the definition $t_1 = \sup\{t \in [0, t_2] : A_k(t) = 0\}$, and the proof will thus be accomplished.

An immediate consequence of the previous theorem is the following uniqueness criterion.

Theorem 4.1.2 . *Let the abstract function $f(t, x)$ be continuous and satisfy $f(t, 0) = 0$ for every $t \in [0, 1]$. Assume further that there exist $\varepsilon > 0$, duality mapping $k : E \to 2^{E^*}$ and $t_0 \in (0, 1]$ such that $u = 0$ is the only root of the auxiliary scalar equation (4.5) with $\|u\| < \varepsilon$ for every $t \in [0, t_0]$. Then the (4.2) admits in the interval $[0, t_0]$ only the zero solution.*

As it was pointed out by Majorana the crucial point in theorems 4.1.1, 4.1.2 is the assumption that (4.2) has the zero solution. We follow Majorana's procedure to remove this restriction. If we know a solution φ of (4.2), then, by means of change of variables $x = p + \varphi(t)$ (4.2) becomes

$$\begin{cases} p' = F(t, p), \\ p(0) = 0, \end{cases} \tag{4.7}$$

where $F(t, p) = f(t, p+\varphi(t)) - f(t, \varphi(t))$. It is clear that any two different solutions of (4.2) are mapped to different solutions of (4.7). Moreover (4.7) admit the zero solution, which corresponds to the solution φ of (4.1). We thus get the counterpart to (4.5).

$$Re < F(t, u), k(u) >= \frac{1}{2t}\|u\|^2. \tag{4.8}$$

We can restate theorems 4.1.1, 4.1.2 involving the nontrivial solution φ instead of $x = 0$.

Theorem 4.1.3 . *Let the abstract function $f(t, x)$ be continuous on $[0, 1] \times E$ and let φ be a solution of (4.2). Further let (4.2) admit two different solutions defined in $[0, a](a < 1)$. Then, for every $\varepsilon > 0$, and for every duality mapping $k : E \to 2^{E^*}$, there exists $t \in [0, a]$ such that (4.8) has at least two different roots u with $\|u\| < \varepsilon$.*

Theorem 4.1.4 . *Let the abstract function $f(t, x)$ be continuous and let φ be a solution of (4.2). Assume further that there exist $\varepsilon > 0$, duality mapping $k : E \to 2^{E^*}$ and $t_0 \in (0, 1]$ such that $u = 0$ is the only root of the auxiliary scalar equation (4.8) with $\|u\| < \varepsilon$ for every $t \in [0, t_0]$. Then the (4.2) admits in the interval $[0, t_0]$ only the solution φ.*

Remark 4.1.1 *Going back to the concrete Example 4.0.1 showing that Majorana's results are not directly extendable to the simple space $\mathbb{R}^2$, we shall here apply Theorem 4.1.1. For this purpose some concepts are required to be emphasized. It is well known that the following statements are equivalent [33]:*

(i) *E is convex,*

(ii) *E is smooth (i.e., for every $x \in E$ there is a unique continuous functional x^* such that*

$< x^*, x >= ||x||$ and $||x^*|| = 1$),

(iii) the duality mapping is a single-valued.

It is easy to see that the scalar equation (4.1) will take the form:

$$\frac{2u_1}{\sqrt{\|u\|}}(u_1 + u_2) + \frac{2u_2}{\sqrt{\|u\|}}(u_2 - u_1) = \frac{u_1^2 + u_2^2}{t}.$$

Clearly, $u = 0$ corresponds to the trivial solution $x = 0$, and by straightforward calculations, one can show that $u = x(2t_2)$ is the second root corresponding to the nontrivial solution, where $x(t_2) \neq 0$ and t_2 is specified in the proof of Theorem 4.1.1.

4.2 Uniqueness Results for Ordinary Differential Equations in Infinite Dimensional Banach Spaces

This section contains our main second result in which we provide an abstract version of Majorana' s uniqueness theorem for the Cauchy problem

$$x' = f(t, x), \qquad x(0) = 0, \tag{4.9}$$

where $f : [0, a] \times E \to E$, E is an infinite dimension real Banach space endowed with weak topology. We then place (4.1) with a another one which is based on the following definition.

Definition 4.2.1 *[63]. Let E be a real Banach space. A subset P of E is called a cone if the following are true:*

(i) P is nonempty and nontrivial (i.e. P contains a nonzero point);

(ii) $\lambda P \subset P$, $\lambda > 0$;

(iii) $P + P \subset P$;

(iv) $\overline{P} = P$, where $\overline{P}$ denotes the closure of P;

(v) $P \cap -P = 0$, where, 0 denotes the zero element of the Banach space E.

Assume that $P^0 \neq \phi$, P^0 denotes the interior of P.

Definition 4.2.2 *[63]. The cone P of E induces an ordering " $\leq$ " by setting:*

$$x \leq y \Leftrightarrow y - x \in P,$$

$$x < y \Leftrightarrow y - x \in P^0.$$

Let P^* be the set of all continuous linear functionals c on E such that $c(x) \geq 0$ for all $x \in P$ and let P_0^* be the set of all continuous linear functionals c on E such that $c(x) > 0$ for all $x \in P^0$. The construction of the counterpart to (2.2) is based, among other, on the following Lemma which is due to Mazur [68].

Lemma 4.2.1 *[63]. Let P be a cone with nonempty interior P^0. then the following hold:*
(i) $x \in P$ is equivalent to $c(x) \geq 0$ for all $c \in P^*$;
(ii) $x \in \partial P$ implies that there exists a $c \in P_0^*$ such that $c(x) = 0$, where, ∂P denotes the boundary of P.

We concentrate ourself to weak solutions, then the classical case in the subject is that due to Szep [89]. Before giving Szep's theorem we need the following definition.

Definition 4.2.3 *[63]. A function $f(t, x)$ is said to be weakly-weakly continuous at (s, y) if given $\varepsilon > 0$ and $\varphi \in E^*$ there exist $\delta(\varepsilon, \varphi) > 0$ and $\gimel(\varepsilon, \varphi)$ a weakly open set containing y such that $|\varphi(f(t, x) - f(s, y))| < \varepsilon$ whenever $|t - s| < \delta$ and $x \in \gimel$.*

Definition 4.2.4 . *A function $x(t)$ is said to be weakly differentiable at t_0, if there exists a point in E denoted by $x'(t_0)$ such that $\varphi(x'(t_0)) = (\varphi x)'(t_0)$ for every $\varphi \in E^*$.*

Definition 4.2.5 . *A solution of (4.9) on $[0, a]$, $a < 1$ is a function x defined on $[0, a]$ such that $x(t)$ is weakly continuous, and for almost every $t \in [0, a]$, $Dx(t) = f(t, x(t))$ where $Dx(t)$ is the weak limit $\lim_{h \to 0} h^{-1}(x(t + h) - x(t))$.*

Theorem 4.2.1 *[89]. Let E be a reflexive Banach space and f be a weakly-weakly continuous function on $A : 0 \leq t \leq a, \|x\| \leq b$. Let $\|f(t, x)\| \leq M$ on A. Then (4.9) has at least one weak solution defined on $[0, \alpha]$, $\alpha = min\{a, \frac{b}{M}\}$.*

Theorem 4.2.2 . *Let Szep's hypotheses hold and let $f(t,0) = 0$ for every $t \in [0,1]$. Suppose further that (4.9) admit two different weak solutions defined in $[0,\alpha](\alpha < a)$. Then, for every $\varepsilon > 0$, and every $c \in P_0^*$, there exists $t \in [0,\alpha]$ such that the following scalar equation*

$$c(u) = tc(f(t,u)), \tag{4.10}$$

has at least two different roots u with $\|u\| < \varepsilon$.

Following the same pattern as that used by [65].

Proof. *It follows, by the assumption $f(t,0) = 0$, that (4.9) has the zero solution, so we assume that (4.9) has the weak solution $y(t) \neq 0$.*

Let $\varepsilon > 0$ and $c \in P_0^*$ be given. Since $u = 0$ is a root of (4.10) for every $t \in [0,a]$, it is sufficient to show that there exists $t \in [0,\alpha]$ for which (4.10) is satisfied by some $u \neq 0$ with $\|u\| < \varepsilon$. Let a real-valued function $A_c(t)$ be defined by setting

$$A_c(t) = \begin{cases} \frac{c(y)}{t} & t \neq 0 \\ 0 & t = 0, \end{cases}$$

$t \in [0,a]$. Of course $A_c(t)$ is nonnegative weak continuous in $[0,a]$, weakly differentiable in $(0,a)$, and for every t in $(0,a)$ we have

$$A_c'(t) = \frac{1}{t^2}[tc(f(t,y(t))) - c(y)]. \tag{4.11}$$

Now, fix $t_2 \in (0,a)$ with $y(t_2) \neq 0$ such that $\|y(t)\| < \varepsilon$ for every $t \in [0,t_2]$. Denote $t_1 = \sup\{t \in [0,t_2] : A_c(t) = 0\}$. Clearly $y(t_1) = 0$ and $y(t) \neq 0$ for every $t \in (t_1,t_2]$. At this point there are just two possibilities:

P1: If there exists an $t \in (t_1,t_2]$ such that $A_c'(t) = 0$, then from (4.11) for such a t, (4.10) is satisfied by $u = y(t)$. Hence the proof is accomplished just taking these t and $u = y(t)$.

P2: Otherwise if $A_c'(t) \neq 0$ for every $t \in (t_1,t_2]$. According to Darboux property $A_c'(t)$ has a constant sign in $(t_1,t_2]$, then we take $u = y(t_2)(\neq 0)$, and define

$$G(t) = tc(f(t,y(t_2))) - c(y(t_2)), \quad for\ every\ t \in [0,a].$$

Now let us suppose that $A_c'(t) > 0$ for every $t \in (t_1,t_2]$. $G(0) = -c(y(t_2)) < 0$. On the other hand, we have $G(t_2) = t_2^2 A_c'(t_2) > 0$. It follows, by continuity of G, that there exists an $t \in (0,t_2]$

such that $G(t) = 0$. The fact that $A_c(t) > 0$ for every $t \in (t_1, t_2]$ and $A_c(0) = 0$ implies that $A_c'(t) < 0$ is impossible and the proof will thus be accomplished.

An immediate consequence of theorem 4.2.8 is the following uniqueness criterion.

Theorem 4.2.3 . *Let Szep's hypotheses hold and let $f(t, 0) = 0$ for every $t \in [0, 1]$. Assume further that there exist $\varepsilon > 0$, $c \in P_0^*$, and $t_0 \in (0, 1]$ such that $u = 0$ is the only root of the auxiliary scalar equation (4.10) with $\|u\| < \varepsilon$ for every $t \in [0, t_0]$. Then the (4.9) admits in the interval $[0, t_0]$ only the zero solution.*

As it was pointed out by Majorana the crucial point in Theorems 4.2.8, 4.2.9 is the assumption that (4.9) has the zero solution. We follow Majorana's procedure to remove this restriction. If we know a weak solution y of (4.9), then, by means of change of variables $x = p + y(t)$ (4.9) becomes

$$\begin{cases} p' = F(t, p), \\ p(0) = 0, \end{cases} \tag{4.12}$$

where $F(t, p) = f(t, p + y(t)) - f(t, y(t))$. It is clear that any two different weak solutions of (4.9) are mapped to different weak solutions of (4.12). Moreover (4.12) admit the zero solution 0, which corresponds to the solution y of (4.9). We thus get the counterpart to (4.10).

$$c(u) = tc(F(t, u)). \tag{4.13}$$

We can restate theorems 4.2.8, 4.2.9 involving the nontrivial solution y instead of $x = 0$.

Theorem 4.2.4 . *Let Szep's hypotheses hold and let y be a weak solution of (4.9). Further let (4.9) admit two different weak solutions defined in $[0, \alpha](\alpha < a)$. Then, for every $\varepsilon > 0$, and for every $c \in P_0^*$, there exists $t \in [0, \alpha]$ such that (4.13) has at least two different roots u with $\|u\| < \varepsilon$.*

Theorem 4.2.5 . *Let Szep's hypotheses hold and let* y *be a weak solution of (4.9). Assume further that there exist* $\varepsilon > 0$*,* $c \in P_0^*$*, and* $t_0 \in (0, 1]$ *such that* $u = 0$ *is the only root of the auxiliary scalar equation (4.13) with* $\|u\| < \varepsilon$ *for every* $t \in [0, t_0]$*. Then the (4.9) admits in the interval* $[0, t_0]$ *only the weak solution* y*.*

Conclusion. We have taken the time-interval to be $[0, 1]$ and the initial value $x(0)$ to be 0 only for simplicity of notation; our argument would work just as well for any other compact interval and any other initial value. The problem of existence of solutions is not treated herein. Szep's assumptions can be replaced by any set of sufficient conditions that guarantee existence of solutions for (4.9) in reflexive as well as in non-reflexive Banach spaces. This presents an up-to-date account of the advances made in the study of the abstract Cauchy problem and will hopefully stimulate further research in this area.

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